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Finding Infinitesimal Motions of Objects in Assemblies Using Grassmann-Cayley Algebra

E. Staffetti, L. Ros, and F. Thomas
Institut de Robòtica i Informàtica Industrial (UPC-CSIC)
Gran Capità 2-4, 08034 Barcelona, Spain

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1 Introduction

We present a method for deriving automatically the set of allowed infinitesimal motions of a polyhedron in contact with a polyhedral assembly without breaking the established basic contacts. The result is obtained, under the frictionless assumption, by describing each basic contact by means of the Grassmann-Cayley algebra and using cycle conditions over closed kinematic chains between the polyhedron and the assembly. Although, in practice, assemblies need to be separated completely and not only infinitesimally, this constitute a very useful information for an assembly sequence planner [Thomas et al. 1992], [Staffetti et al. 1998]. The proposed technique is also applied to solve infinitesimal mobility analysis problems of general multiloop spatial mechanisms.

2 Background

In this section we give a general overview of the part of the Grassmann-Caley algebra needed in subsequent developments without going into mathematical details. A deeped insight can be found in [White 94] or [White 95].

2.1 Projective Space and Plücker Coordinates

Let us consider the projective 3-space. A point q in this space is represented by a non-zero 4-tuple $q = (q_1, q_3, q_3, q_4)$ whose elements are called the homogeneous coordinates of the point. Two 4-tuples p and q represent the same projective point if, and only if, $p = \lambda q$ for some $\lambda \neq 0$. If $q_4 \neq 0$ we say the point is finite and it can be represented by the 4-tuple $p = (p_1, p_2, p_3, 1)$ where the first three components are the Euclidean coordinates of the same point indicated with $\mathbf{p}$. If $q_4 = 0$ the point lies on the plane at infinity.

Given two points a and b in homogeneous coordinates a line L through them can be represented by the vector P_L formed by the six 2×2 minors of the following 2×4 matrix:

$$\begin{pmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \end{pmatrix}$$

called the Plücker coordinates of the line. It can be proven that:

$$P_L = \begin{pmatrix} b_1 - a_1 & b_2 - a_2 & b_3 - a_3 & a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \end{pmatrix} = (\mathbf{s}, \mathbf{r} \times \mathbf{s})$$

where $\mathbf{s} = (\mathbf{b} - \mathbf{a})$ and $\mathbf{r}$ is the Euclidean position of any point on L. This operations corresponds to the exterior product of the Cayley algebra, a modern version of the Grassmann algebra [Dubilet et al. 1974], [White 1995]. In this algebra the subspace generated by a and b is called the 2-extensor of a and b and its symbolic Plücker coordinates are indicated by $a \vee b$. Thus the line L can be expressed as $L = a \vee b$.

The point at infinity on L is the vector

$$\begin{pmatrix} b_1 - a_1 & b_2 - a_2 & b_3 - a_3 & 0 \end{pmatrix}.$$

Each 4-tuple of the form $(t_1, t_2, t_3, 0) \neq (0, 0, 0, 0)$ represents a point at infinity. This point can be thought as infinitely far away in the direction given by $\mathbf{s}$. The same point at infinity lies on every line parallel to L but non-parallel lines have distinct points at infinity.

A line at infinity is determined by two distinct points at infinity:

$$\begin{pmatrix} s_1 & s_2 & s_3 & 0 \\ t_1 & t_2 & t_3 & 0 \end{pmatrix},$$

which has the following vector of Plücker coordinates

$$P_L = \begin{pmatrix} 0 & 0 & 0 & s_2t_3 - s_3t_2 & s_3t_1 - s_1t_3 & s_1t_2 - s_2t_1 \end{pmatrix}.$$

Likewise, the plane P determined by the three points a, b, and c is a 3-extensor indicated by $a \vee b \vee c$ whose Plücker coordinates are the four 3×3 minors of the following 3×4 matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \end{pmatrix}.$$

It can be easily proven that the Plücker coordinates vector of the plane P can be ordered such that $P_P = (\mathbf{n}, -\mathbf{r} \cdot \mathbf{n})$, where $\mathbf{n}$ is the normal vector to P and $\mathbf{r}$ is the Euclidean position vector of any point on P.

2.2 Projective Representation of Motions

Let $\mathbf{u}$ be the Euclidean velocity of an Euclidean point $\mathbf{p}$. The motion of the projective point p can be defined as $M(p) = (\mathbf{u}, -\mathbf{u} \cdot \mathbf{p})$, that is, the 3-extensor that represents the plane through the point $\mathbf{p}$ perpendicular to $\mathbf{u}$.

An instantaneous motion in projective 3-space, that is an assignment of motions $M(p_i)$ to the projective points p_i, is a rigid motion if the velocities preserve all distances in space. In projective terms rigid motions can be expressed in a simple and effective way [White 1994].

If r and s are projective points, for each point p in space we define $M(p) = r \vee s \vee p$. This assignment of motion preserves all distances and therefore it will correspond to a rigid motion in space determined by the 2-extensor $C = r \vee s$. This 2-extensor, that represents the line through r and s, is called the center of the motion. Since $M(r) = 0$ and $M(s) = 0$, this represents a rotation around the axis determined by r and s.

A translation can be described as a rotation about an axis at infinity. Let $a = (a_1, a_2, a_3, 0)$ and $b = (b_1, b_2, b_3, 0)$ be two points at infinity. Then, the extensor $a \vee b$ can be used as the center of a motion $M(p) = a \vee b \vee p$. The corresponding velocity is $\mathbf{v} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$. Since it is independent from the point p, it will represent a translation.

By composing translations and rotations represented by its centers C_i a more general screw motion in space can be obtained. Instantaneously, this composition corresponds to a simple addition of the motion centers C_i, that is, the equivalent motion is $M(p) = \sum_i (C_i \vee p) = (\sum_i C_i) \vee p$.

2.3 GR Graphs

Let us consider two bodies hinged along the line $a \vee b$ then, for any instantaneous motion of the bodies with centers S_1 and S_2, there is a scalar λ such that $S_1 - S_2 = \lambda(a \vee b)$. This concept can be extended to any number of rigid bodies and hinges [Crapo et al. 1982].

An articulated structure is a set (B, A) where B is a finite collection of bodies $(B_1, \ldots, B_n)$, and $A = (\ldots, L_{i,j}, \ldots)$ is a set of hinges represented by non zero 2-extensors in projective space indexed by ordered pair of indices with $L_{i,j} = -L_{j,i}$. An instantaneous motion of the articulated structure (B, A) is an assignment of a center S_i to each body B_i such that for each hinge $L_{i,j} \in A$ and for some choice of scalars $\omega_{i,j}$ we obtain $S_i - S_j = \omega_{i,j} L_{i,j}$. The scalars $\omega_{i,j}$ being rotational or translational velocities at the hinges $L_{i,j}$.

In general, we accept that the same boby appears in different articulated structures. This situation can be easily represented using a directed graph – called GR graph – whose nodes will represent bodies and, if body B_i is restricted in its motion with reference to body B_j, there will be a directed arc going from node B_i to node B_j labeled with $\omega_{i,j} L_{i,j}$.

Now, consider that a GR graph has a cycle, for example B_0, $\omega_{0,1} L_{0,1}$, B_1, $\omega_{1,2} L_{1,2}$, B_2, $\ldots$, B_k, $\omega_{k,0} L_{k,0}$, B_0. Since the net velocity around a cycle must be zero, we obtain the following loop equation:

$$\omega_{0,1} L_{0,1} + \omega_{1,2} L_{1,2} + \cdots + \omega_{k,0} L_{k,0} = 0. \tag{1}$$

which constraints velocities $\omega_{0,1}$, $\omega_{1,2}$, $\ldots$, $\omega_{k,0}$. This can be done for any cycle in a GR graph. As a consequence, velocities $\omega_{i,j}$ must satisfy the set of loop equations resulting from all possible cycles in the graph.

If for an arbitrary GR graph we obtain the loop equations resulting from all possible cycles, we would find that many equations are linearly dependent from the others. This is reason why, in practice, we only need to consider the loop equations resulting from a complete set of basic cycles [Thomas 1992].

Example 1. Let us consider the single loop mechanism shown in *fig. 1a...*

(a) (b)

Figure 1. A single loop mechanism (a), and its associated GR graph (b).

3 Motion of Polyhedra in Contact

(a) (b)

Figure 2. Infinitesimal motions that keep a type-A contact (a), and a type-B contact (b) between two polyhedra.

Any contact between polyhedra can be expressed as the composition of two basic contacts; namely: type-A and type-B contacts. A type-A contact occurs when a vertex v of a polyhedron touches a face the other polyhedron, and a type-B contact occurs when an edge of one polyhedron in contact with an edge of other polyhedron (see, for example, [Thomas 1994]).

The motion constraints between two polyhedral bodies B_i and B_j under a type-A contact can be thought as produced by a spherical and a planar joint between them. The spherical joint can be modelled as three revolute joints whose axes intersect in the contact point and the planar joint can be described with two prismatic joints that permit translations along two non-parallel axes on the plane of the face. In this case S_i and S_j, the centers of the motion of B_i and B_j, respectively, are related by the following expression:

$$S_i = S_j + \omega_{i,j}^{A,r_1} L_{i,j}^{A,r_1} + \omega_{i,j}^{A,r_2} L_{i,j}^{A,r_2} + \omega_{i,j}^{A,r_3} L_{i,j}^{A,r_3} + \omega_{i,j}^{A,t_1} L_{i,j}^{A,t_1} + \omega_{i,j}^{A,t_2} L_{i,j}^{A,t_2}, \tag{2}$$

where $L_{i,j}^{A,r_1}$, $L_{i,j}^{A,r_2}$, $L_{i,j}^{A,r_3}$ are 2-extensors that define the centers of rotation and $L_{i,j}^{A,t_1}$ and $L_{i,j}^{A,t_2}$ are the axes at infinity that represent translations on the face plane (*fig. 2a*).

Likewise, the motion of two polyhedral bodies B_i and B_j, constrained to maintain a B-type contact, can be modelled by means of a composition of two cylindrical and one revolute joints whose axes intersect in the contact point. Thus, we obtain the following relation between the centers of motion:

$$S_i = S_j + \omega_{i,j}^{B,r_1} L_{i,j}^{B,r_1} + \omega_{i,j}^{B,t_1} L_{i,j}^{B,t_1} + \omega_{i,j}^{B,r_2} L_{i,j}^{B,r_2} + \omega_{i,j}^{B,t_2} L_{i,j}^{B,t_2} + \omega_{i,j}^{B,r_3} L_{i,j}^{B,r_3} \quad (3)$$

where $L_{i,j}^{B,r_1}$ and $L_{i,j}^{B,r_2}$ are the 2-extensors that define rotations around the edges in contact, whereas $L_{i,j}^{B,t_1}$ and $L_{i,j}^{B,t_2}$ are 2-extensors that define translations along directions parallel to each edge. $L_{i,j}^{B,r_3}$ represents a rotation axis normal to the plane that contains the edges in contact (*fig. 2b*).

(a) **(b)**

Figure 3. Two polyhedra in contact along an edge (a), and the equivalent representation in terms of four basic contacts expressed in terms of a GR graph (b).

Example 2. Let us consider the two polyhedra in contact appearing in *fig. 3a.* They are in contact along an edge. This contact can be expressed in terms of four basic contacts ...

(a) **(b)**

Figure 4. Four workpieces to be assembled (a), and the associated motion in contact constrains expressed as a GR graph (b).

Example 3. Now, consider the four workpieces to assembled shown in *fig. 4* ....

4 Conclusions

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