A Group-Theoretic Approach to the Computation of Symbolic Part Relations

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Abstract—**When a set of constraints is imposed on the degrees of freedom (d.o.f.) between several rigid bodies, finding the configuration or configurations that satisfy all these constraints is a matter of special interest. The problem is not new and has been discussed, not only in kinematics, but also more recently in the design of object level robot programming languages. In this last domain, several languages have been developed, from different points of view, that are able to partially solve the problem. A more general method than those previously proposed based on the symbolic manipulation of chains of matrix products is derived using the Theory of Continuous Groups.**

I. Introduction

A TRUE AUTOMATION of design and manufacturing processes will not be attained until man–machine interaction is drastically reduced and simplified, as compared to current standards. A high-level object level robot programming language is of no use if the burden of specifying the object models and their spatial relationships outweights the effort of programming robot motions in detail.

In the assembly domain, it does not suffice to make the workpiece models (possibly produced by a CAD system) available in the programming environment, but a description of the way the different pieces should be fitted together is also required. This description can be provided in full detail by either the designer or the programmer, or be automatically inferred from constraints derived from both the shapes of the workpieces involved in the assembly and the mechanics of the assembly operations themselves.

The method we have developed consists of propagation, combination, and satisfaction of three types of constraints: shape-matching constraints between the mating parts of workpieces, constraints on the degrees of freedom (d.o.f.) between workpieces, and nonintersection constraints [16]. In this paper, we will confine ourselves to the detailed description of the procedure developed to deal with constraints on the d.o.f.

Let us look at an example that illustrates the sorts of problems to be solved. According to Fig. 1 and imposing that face P_1 be against P_4 and P_2 against P_3, is there any configuration satisfying both constraints? If the answer is yes, how many d.o.f. remain between the box and the cube? Which are they? Which are the values of the constrained d.o.f.?

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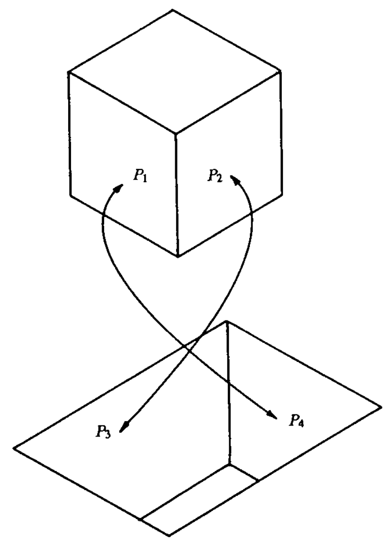

Fig. 1. Example used to illustrate the sorts of problems to be solved.

Previous work to solve these problems within the robotics domain has been carried out by Taylor [15], Popplestone and his colleagues [2], [3], [9], [12], [13], and Mazer [10].

Taylor deals with both constraints on the d.o.f. (equalities) and nonintersection constraints (inequalities), and applies a linear programming methodology to solve the resulting equations numerically. Because of the linearization procedure used, his method can only deal with one rotational d.o.f. that is confined to envolve within the union of narrow ranges. Taylor himself points out that his method is better suited for tackling problems involving only small inaccuracies in the positioning of objects, rather than those involving complete indeterminacy along several d.o.f. As possible applications, he suggests the determination of the most critical tolerances between workpieces during the design phase and the construction of appropriate fixtures.

Popplestone and his colleagues have only dealt with constraints on the d.o.f. They have explored two methodologies for chaining and merging constraints of this type: A system of rewriting rules [12] and a table look-up procedure [3], [9]. The slowness of the former methodology led to the development of the second one, whose worst shortcoming derives from the non-closeness of the set of constraints on the d.o.f. considered under the composition operation and, thus, not all combinations of constraints can be tabulated. Their procedures have been implemented as part of the RAPT

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interpreter—an object level robot programming system—and their aim has not been to infer feasible assembly configurations, but to derive the relative positions and orientations of the workpieces and the robot through the different stages in the performance of a task.

Mazer [10] has followed a different approach in the development of the LM-Geo interpreter—another object level robot programming system. The main difference between his and the former systems is the use of vector equations as description of the problem.

This paper is structured as follows. In Section II the notion of displacement is introduced, as well as some of its basic properties and the notation used throughout this paper. In Section III, a classification of the subgroups of displacements into conjugation classes is presented allowing us to represent—in Section IV—spatial relationships as compositions and intersections of subgroups, an idea already sketched by Popplestone [13]. Finally, a simple symbolic procedure to obtain numerical values for the constrained d.o.f. is given in Section V.

II. Homogeneous Transformations: Generalities and Notation

The representation of objects in an n-dimensional space using homogeneous coordinates needs a space of dimension $n + 1$ from which the original space is recovered by projection. For example, the vector $\boldsymbol{v} = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$, where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are unit vectors along the Cartesian coordinate axes, is represented using homogeneous coordinates as a column vector

$$\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

so that

$$x_1 = x/t$$

$$y_1 = y/t$$

$$z_1 = z/t.$$

Henceforth we will normalize $t = 1$.

A transformation $\boldsymbol{H}$ is a 4×4 matrix so that, the image of a given point $\boldsymbol{v}$ under this transformation is represented by the matrix product $\boldsymbol{u} = \boldsymbol{Hv}$.

A. Translations

A transformation $\boldsymbol{H}$ representing a translation by a vector $\boldsymbol{d} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ will be

$$\boldsymbol{H} = \textbf{Trans}\ (\boldsymbol{d}) = \textbf{Trans}\ (a, b, c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus given a vector $\boldsymbol{v} = (x, y, z, 1)^t$, its image $\boldsymbol{u}$ under $\boldsymbol{H}$ will be

$$\boldsymbol{u} = \boldsymbol{Hv} = \begin{bmatrix} x+a \\ y+b \\ z+c \\ 1 \end{bmatrix}.$$

It is easy to demonstrate that the set of all translations constitutes a group under the matrix product operation, which will be denoted by T.

B. Rotations

The transformations representing rotations about the x, y, and z axes by angles ψ, θ, or ϕ, respectively, are

$$\textbf{Rot}\ (x, \psi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\textbf{Rot}\ (y, \theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\textbf{Rot}\ (z, \phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Each element ij of the 3×3 upper left submatrix is equal to the cosine of the angle between the i-axis of the original coordinate frame and the j-axis of the rotated one.

These matrices, as well as their products, are orthogonal matrices with determinants equal to $+1$. They also constitute a group under matrix multiplication which will be denoted by S_o.

C. Displacements

The transformations representing rotations and translations can be composed and the resulting matrices are said to describe displacements. The set D of all displacements has the characteristic properties of a continuous group of dimension 6.

The following properties must be emphasized:

- *Decomposition of a displacement:* Every displacement $\boldsymbol{H}$ can be decomposed into the product of a translation and a rotation, so that

$$\boldsymbol{H} = \textbf{Trans}\ (\boldsymbol{d})\hat{\boldsymbol{H}} = \textbf{Trans}\ (a, b, c)\hat{\boldsymbol{H}}, \qquad \forall \boldsymbol{H} \in D \quad (1)$$

where $\hat{\boldsymbol{H}}$ is the rotation component of the displacement $\boldsymbol{H}$ or, in other words, is the matrix resulting from setting the first three elements—a, b, and c—of the last column of $\boldsymbol{H}$ to zero.

- *Composition of n displacements:*

$$\begin{aligned} &\boldsymbol{H}_1 \cdots \boldsymbol{H}_i \cdots \boldsymbol{H}_n \\ &\quad = \textbf{Trans}\ (\boldsymbol{d}_1)\hat{\boldsymbol{H}}_1 \cdots \textbf{Trans}\ (\boldsymbol{d}_n)\hat{\boldsymbol{H}}_n \\ &\quad = \textbf{Trans}\ (\boldsymbol{d}_1 + \hat{\boldsymbol{H}}_1\boldsymbol{d}_2 + \cdots + \hat{\boldsymbol{H}}_1\hat{\boldsymbol{H}}_2 \cdots \hat{\boldsymbol{H}}_{n-1}\boldsymbol{d}_n) \\ &\quad\quad \cdot \hat{\boldsymbol{H}}_1\hat{\boldsymbol{H}}_2 \cdots \hat{\boldsymbol{H}}_n, \qquad \forall \boldsymbol{H}_1 \cdots \boldsymbol{H}_n \in D. \end{aligned} \quad (2)$$

TABLE I
CLASSIFICATION OF THE SUBGROUPS OF D INTO CONJUGATION CLASSES

Dimension (d.o.f.)	Notation	Constraint	Geometric Elements of Definition	Direct and Inverse Symbolic Displacements	Associated Lower Pair
1	T_v	rectilinear translation	a direction of translation given by a vector v	**Trans** $(x, 0, 0)$ **Trans** $(-x, 0, 0)$	(P) Prismatic
	R_u	rotation around an axis	an axis of revolution u	**Twix** (ψ) **Twix** $(-\psi)$	(R) Revolution
	$H_{u,p}$	helicoidal movement	an axis of revolution u and a thread pitch p	**Trans** $(x, 0, 0)$ **Twix** (px) **Trans** $(-x, 0, 0)$ **Twix** $(-px)$	(H) Screw
2	T_P	planar translation	a plane P	**Trans** $(0, y, z)$ **Trans** $(0, -y, -z)$	
	C_u	lock movement	an axis u	**Trans** $(x, 0, 0)$ **Twix** (ψ) **Trans** $(-x, 0, 0)$ **Twix** $(-\psi)$	(C) Cylindrical
3	T	spatial translation		**Trans** (x, y, z) **Trans** $(-x, -y, -z)$	
	G_P	planar sliding	a plane P	**Trans** $(0, y, z)$ **Twix** (ψ) **Trans** $(0, -y\cos\psi - z\sin\psi, y\sin\psi - z\cos\psi)$ **Twix** $(-\psi)$	(E) Plane
	S_o	spheric rotation	a point o in the space	**Twix** (ψ) **XTOY Twix** (ξ) **XTOY Twix** (η) **Twix** $(-\eta - \pi)$ **XTOY Twix** $(-\xi)$ **XTOY Twix** $(-\psi)$	(S) Spherical
	$Y_{v,p}$	Y movement	a direction of revolution v and a thread pitch p	**Trans** (x, y, z) **Twix** (px) **Trans** $(-x, -y\cos\psi - z\sin\psi, y\sin\psi - z\cos\psi)$ **Twix** $(-px)$	
4	X_v	X movement	a direction of revolution v	**Trans** (x, y, z) **Twix** (ψ) **Trans** $(-x, -y\cos\psi - z\sin\psi, y\sin\psi - z\cos\psi)$ **Twix** $(-\psi)$	

Since $\hat{H}_1$, $\hat{H}_2$, $\cdots$ and $\hat{H}_n$ belong to S_o, their product still represents a spheric rotation.

If a transformation is postmultiplied by another transformation, the latter is applied with respect to the transformed frame described by the former. Conversely, if a transformation is premultiplied by another one, the latter is applied with respect to the reference frame [11]. Other authors [12], in using the transposes of the above defined transformations, adhere to the inverse rule.

• *Inverse displacement:* Because of the properties of orthogonal matrices, the inverse displacement of $\boldsymbol{H}$ is

$$H^{-1} = \hat{H}^t \,\textbf{Trans}\,(-a, -b, -c), \qquad \forall H \in D \tag{3}$$

where $\hat{H}^t$ denotes the transpose matrix of $\hat{H}$.

D. Symbolic Operators

Let us define three symbolic operators [2] that will allow us to describe any displacement.

- **Trans** (a, b, c). As it was stated above.
- **Twix** (ψ). It is equivalent to transformation **Rot** (x, ψ).
- **XTOY**. This operator rotates the x-axis in such a way that it becomes the y-axis. It is equivalent to **Rot** $(z, \pi/2)$.

Any displacement can be described using only these three operators. Actually,

$$H = \textbf{Trans}\,(a, b, c)\ \textbf{Twix}\,(\psi)\ \textbf{XTOY Twix}\,(\phi) \cdot \textbf{XTOY Twix}\,(\theta), \qquad \forall H \in D. \tag{4}$$

If the vector $(1, 0, 0, 0)^t$ is an eigenvector of $\hat{H}$, there is no single solution for ψ and θ (see Section V-B3). Then we can take, as a convention, $\theta = 0$. Thus for example, we have

$$\textbf{Rot}\,(y, \pi) = \textbf{Twix}\,(\pi)\ \textbf{XTOY XTOY}. \tag{5}$$

III. SUBGROUPS OF THE GROUP OF DISPLACEMENTS AND CONSTRAINTS ON THE d.o.f

A *group* is a set of elements closed under an associative operation with an identity and inverse elements, as is the group D of displacements. A *subgroup* $S \subset D$ is a subset of D which is itself a group under the same operation. The composition of elements of D can be extended to the composition of elements and subgroups. If $S \subseteq D$ and $\boldsymbol{D} \in D$, then the *right coset* $S \cdot \boldsymbol{D}$ is the set $\{\boldsymbol{H}\cdot\boldsymbol{D} \mid \boldsymbol{H} \in S\}$. The *left coset* $\boldsymbol{D}\cdot S$ and the *two-sided coset* $\boldsymbol{D}_1 \cdot S \cdot \boldsymbol{D}_2$ can be similarly defined [8]. More generally, the composition of two subgroups $S_1 \cdot S_2$ is defined as $\{\boldsymbol{D}_1 \cdot \boldsymbol{D}_2 \mid \boldsymbol{D}_1 \in S_1, \boldsymbol{D}_2 \in S_2\}$.

Definition 1. Conjugation classes of subgroups of D. Every such class is an equivalence class with respect to the relation

$$S_1 \sim S_2 \Leftrightarrow \exists \boldsymbol{D} \in D \mid S_2 = \boldsymbol{D} S_1 \boldsymbol{D}^{-1} \tag{6}$$

S_1 and S_2 being subgroups of D.

An exhaustive classification of the subgroups of D into conjugation classes can be carried out using classic methods of analysis of finite dimension continuous groups [6]. A list of the classes thus obtained is shown in Table I. Note that most of those classes are associated with lower pair couplings [1].[1]

The chains of products of symbolic operators appearing in the table represent the conjugation classes. These chains are obtained, as in the RAPT system, assuming the following conventions for the local coordinate frames of the geometric elements of definition:

[1] When two bodies with possibility of relative movement keep contact along a surface, it is said in kinematics that there exists a lower pair coupling between them.

TABLE II
CONDITIONS OF INCLUSION OF ONE SUBGROUP OF D INTO ANOTHER
(Adapted from Hervé [6].)

	T_u	R_u	$H_{u,p}$	T_P	C_u	T	G_p	S_o	$Y_{v,p}$	X_v
T_P	$u \| P$									
$C_{u'}$	$u \| u'$	$u \bowtie u'$	$u \bowtie u'$							
T	$\forall u$			$\forall P$						
$G_{P'}$	$u \| P'$	$u \perp P'$		$P \| P'$						
S_o		$o \in$ axis u								
$Y_{v,p'}$	$u \perp v$		$u = v, p = p'$	$P \perp v$						
$X_{v'}$	$\forall u$	$u \| v'$	$u \| v'$	$\forall P$	$u \| v'$	yes	$P \perp v'$		$u \| v'$	
D	$\forall u$	$\forall u$	$\forall u, \forall p$	$\forall P$	$\forall u$	yes	$\forall P$	$\forall o$	$\forall v, \forall p$	$\forall v$

$u \perp P \Rightarrow u$ perpendicular to P.
$u' \bowtie u \Rightarrow u'$ and u collinear.
$u \| v \Rightarrow u$ and v parallel.

- The origin of the local coordinate frame of a vertex coincides with the vertex.
- The local coordinate frame of an edge has its x-axis aligned with the edge.
- The local coordinate frame of a plane has its x-axis normal to the plane and its origin on it.

Definition 2. Constraints on the d.o.f. Given a reference frame, we call constraint to the chain of matrix products resulting from premultiplying and postmultiplying the chain representation of a conjugation class by constant displacements.

In this paper we will only consider the limitations of movement between two bodies which can be expressed as a constraint or a product of constraints.

The set of all displacements obtained by assigning values to the variables in a constraint constitutes a subgroup of D belonging to the conjugation class whose chain representation coincides with the constraint up to a premultiplication and postmultiplication by constant displacements. If $\boldsymbol{R}$ is a constraint, we denote the subgroup thus obtained by $\boldsymbol{R}^G$ and refer to it as the *subgroup associated* with $\boldsymbol{R}$. Note that two different constraints can have the same associated subgroup, for instance $(\boldsymbol{R}^{-1})^G = \boldsymbol{R}^G$.

Many of the subgroups appearing in Table I are in turn subgroups of other subgroups appearing in the same table. The conditions of inclusion of one subgroup into another are stated in Table II.

Table III shows the results of the operations of intersection and composition of subgroups, for all the subgroups in Table I whose intersection is neither the identity displacement nor any of the two subgroups. The results of the composition are expressed in regular form, i.e., as products of subgroups whose intersection is the identity displacement.

The composition of subgroups of D is not, in general, commutative, but $S_1 \cdot S_2 = S_2 \cdot S_1$ iff $S_1 \cdot S_2$ is also a subgroup of D. For instance, $R_u \cdot T_v = T_v \cdot R_u = C_u$ iff $v \| u$.

IV. Graph of Spatial Relationships: Operations of Composition and Intersection

In our system, each body is described by means of its parts (volumetric primitives) and elements (faces, edges, and vertices). Thus we have a set of bodies, parts, and elements, as well as the constraints between them, represented by chains of matrix products, constituting what is called a directed graph of spatial relationships (g.s.r.), a hierarchical graph with four levels (Fig. 2).

In order to simplify the treatment, all constraints between either the elements or the parts of the bodies will be translated into constraints between the reference frames of the bodies themselves by premultiplying and postmultiplying by the appropriate displacements. Moreover, since we are only interested in the relationships between the bodies, the reference to the world are eliminated. This simplified directed graph, whose nodes are bodies and whose edges are constraints, will be called a GR graph.

The basic operations to be carried out on the GR graph will be the intersection and the composition of two constraints. Both operations appear schematized in Fig. 3.

A. Composition and Intersection of Constraints

Let us assume a universe of three bodies—$\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$—linked by two constraints—$\boldsymbol{R}_{12}$ and $\boldsymbol{R}_{23}$—which, in general, will have different associated subgroups. We want to find out the dimension of $\boldsymbol{R}_{13} = \boldsymbol{R}_{12} \cdot \boldsymbol{R}_{23}$ or, in other words, the number of d.o.f. of the body $\mathcal{B}_3$ with respect to $\mathcal{B}_1$. Since an element of a group with a subgroup can be expressed as the product of an element of the subgroup by another element, we can write

$$\boldsymbol{R}_{12} = \boldsymbol{P}_{12} \cdot \boldsymbol{M}_{12} \qquad \boldsymbol{R}_{23} = \boldsymbol{M}_{23} \cdot N_{23} \tag{7}$$

where $\boldsymbol{M}_{12}^G$ and $\boldsymbol{M}_{23}^G$ is the same subgroup. Therefore, $\boldsymbol{M}_2^G$, where $\boldsymbol{M}_2 = \boldsymbol{M}_{12}\boldsymbol{M}_{23}$, still is the same subgroup and, if this subgroup has the biggest dimension for which (7) are satisfied, then

$$\boldsymbol{M}_2^G = \boldsymbol{R}_{12}^G \cap \boldsymbol{R}_{23}^G. \tag{8}$$

Moreover

$$\boldsymbol{R}_{13} = \boldsymbol{R}_{12} \cdot \boldsymbol{R}_{23} = \boldsymbol{P}_{12} \cdot \boldsymbol{M}_2 \cdot N_{23}. \tag{9}$$

Definition 3. Independence between constraints. Two constraints are said to be independent if the intersection of the

TABLE III
INTERSECTION AND REGULAR REPRESENTATION FOR THE COMPOSITION OF ALL PAIRS OF SUBGROUPS OF D WHOSE INTERSECTION IS DIFFERENT FROM THE IDENTITY DISPLACEMENT

Groups to be Composed	Conditions on the Geometric Elements	Conditions on the Linking Displacement	Intersection	Regular Representation
$T_P \cdot T_{P'}$			$T_v\ v = P \cap P'$	T
$T_P \cdot G_{P'}$			$T_v\ v = P \cap P'$	$X_v\ v \perp P'$
$G_P \cdot G_{P'}$			$T_v\ v = P \cap P'$	$R_u \cdot T \cdot R_{u'}\ u \perp P\ u' \perp P'$
$Y_{v,p} \cdot T_P$	$v \not\angle P$	$l_{11} \neq \pm 1$	$T_{v'}\ v' \Vert P, v' \perp v$	X_v
$Y_{v,p} \cdot G_P$	$v \not\angle P$	$l_{11} \neq \pm 1$	$T_{v'}\ v' \Vert P, v' \perp v$	$X_v \cdot R_u\ u \perp P$
$Y_{v,p} \cdot Y_{v',p'}$	$v \nparallel v'$	$l_{11} \neq \pm 1$	$T_{v''}\ v'' \perp v, v'' \perp v'$	$R_u \cdot T \cdot R_{u'}\ u \Vert v \vert\ u' \Vert v'$
$Y_{v,p} \cdot C_u$	$u \perp v$	$l_{11} = 0$	T_v	$Y_{v,p} \cdot R_u$
$C_u \cdot C_{u'}$	$u \Vert u'$	$l_{11} = \pm 1$ $l_{24} \neq 0$ or $l_{34} \neq 0$	T_u	$C_u \cdot R_{u'}$
$T_P \cdot C_u$	$u \Vert P$	$l_{11} = 0$	T_u	$T_P \cdot R_u$
$T \cdot C_u$			T_u	$X_v\ v \Vert u$
$G_P \cdot C_u$	$u \Vert P$	$l_{11} = 0$	T_u	$G_P \cdot R_u$
$X_v \cdot C_u$	$u \nparallel v$	$l_{11} \neq \pm 1$	T_u	$X_v \cdot R_u$
$Y_{v,p} \cdot C_u$	$u \Vert v$	$l_{11} = \pm 1$	$H_{v,p}$	X_v
$G_P \cdot C_u$	$u \perp P$	$l_{11} = \pm 1$	R_u	$X_v\ v \perp P$
$S_o \cdot C_u$	$o \in$ axis u	$l_{11} = \pm 1$ $l_{24} = 0$ $l_{34} = 0$	R_u	$S_o \cdot Tu$
$S_o \cdot G_P$			$R_u\ o \in$ axis u, $u \perp P$	$S_o \cdot T_P$
$S_o \cdot X_v$			$R_u\ o \in$ axis u, $u \Vert v$	D
$S_o \cdot S_{o'}$			$R_u\ u = oo'/\vert oo' \vert$	$S_o \cdot R_u \cdot R_{u'}$ $o' \in$ axis u, $o' \in$ axis u'
$Y_{v,p} \cdot Y_{v,p'}$ $(p \neq p')$	$v \Vert v'$	$l_{11} = \pm 1$	$T_P\ P \perp v$	X_v
$Y_{v,p} \cdot X_{v'}$	$v \nparallel v'$	$l_{11} \neq \pm 1$	$T_P\ P \perp v$	$X_v \cdot R_{u'}\ u' \Vert v'$
$G_P \cdot Y_{u',p}$	$v \perp P$	$l_{11} = \pm 1$	T_P	X_v
$G_P \cdot X_v$	$v \not\angle P$	$l_{11} \neq \pm 1$	T_P	$R_u \cdot T \cdot R_{u'}\ u \perp P, u'v$
$G_P \cdot T$			T_P	$X_v\ v \perp P$
$Y_{v,p} \cdot T$			T_P	X_v
$X_v \cdot X_{v'}$	$v \nparallel v'$	$l_{11} \neq \pm 1$	T	$R_u \cdot T \cdot R_{u'}\ u \Vert v, u' \Vert v'$

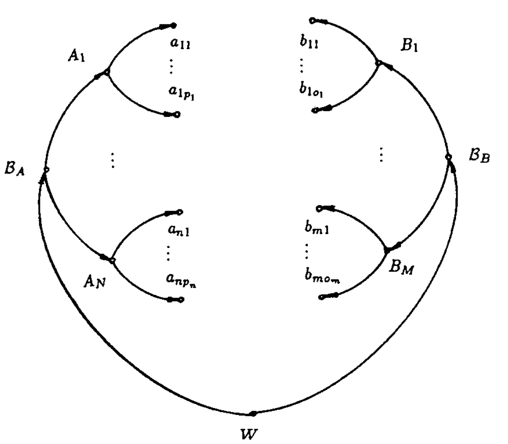

Fig. 2. Graph of spatial relationships (g.s.r.) between two bodies with four levels of hierarchy: World (W), bodies ($\mathcal{B}_A$, $\mathcal{B}_B$), parts ($A_1, A_2, \cdots, A_N$, $B_1, B_2, \cdots, B_M$), and elements ($a_{11}, \cdots, a_{1p_1}, \cdots, b_{m1}, \cdots, b_{mo_m}$). The arrows stand for spatial relationships, not for dependencies.

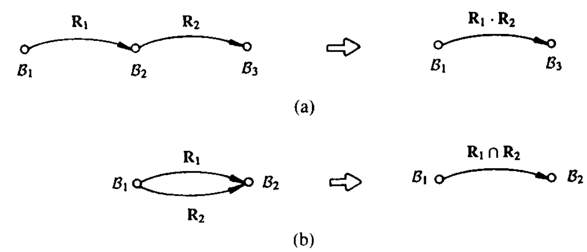

Fig. 3. Operations on the GR graph. (a) Composition of constraints. (b) Intersection of constraints.

subgroups associated with each of them is the identity displacement.

Definition 4. Regular representation. A composition of constraints is said to be regular if it is the composition of independent constraints.

Definition 5. Dimension of a constraint or composition of constraints. The dimension of a constraint or composition of constraints is the number of variables appearing in any of its regular representations.

Notice that (9) is a regular representation for $\boldsymbol{R}_{13}$. It can be also stated that

$$\dim(\boldsymbol{R}_{13}) = \dim(\boldsymbol{P}_{12}) + \dim(\boldsymbol{M}_2) + \dim(\boldsymbol{N}_{23}) \tag{10}$$

and

$$\dim(\boldsymbol{R}_{13}) = \dim(\boldsymbol{R}_{12}) + \dim(\boldsymbol{R}_{23}) - \dim(\boldsymbol{M}_2). \tag{11}$$

If body $\mathcal{B}_3$, still in the same example above, is rigidly linked to $\mathcal{B}_1$ forming a closed kinematic chain, the intermediate body $\mathcal{B}_2$ will only have the possibilities of movement given by $\boldsymbol{R}_{12}^G \cap \boldsymbol{R}_{23}^G$.

B. Examples

Example 1. The composition $G_P \cdot X_v$: The composition of these subgroups, in terms of composition of constraints, can be expressed as

$$\boldsymbol{R}_c = \textbf{Trans}\ (0, y, z)\ \textbf{Twix}\ (\theta)\boldsymbol{L} \cdot \textbf{Trans}\ (x', y', z')\ \textbf{Twix}\ (\psi) \quad (12)$$

where $\boldsymbol{L}$ is the *linking displacement* between both constraints. On the other hand, G_P and X_v can be decomposed into subgroups as follows:

$$\left.\begin{aligned} G_P &= T_P \cdot R_u = R_u \cdot T_P \\ X_v &= T \cdot R_{u'} = R_{u'} \cdot T \end{aligned}\right\} \quad (13)$$

with $\boldsymbol{u} \perp P$ and $\boldsymbol{u}' \parallel v$.

When $\boldsymbol{v} \not\perp P$, $l_{11} \neq \pm 1$ (see Table III) and the only possible simplification of $\boldsymbol{R}_c$ is the following:

$$\boldsymbol{R}_c = \textbf{Trans}\ (x'', y'', z'')\ \textbf{Twix}\ (\theta)\boldsymbol{L}\ \textbf{Twix}\ (\psi). \quad (14)$$

The simplified term, **Trans** $(0, y, z)$, corresponds to the intersection of G_P and X_v. Therefore, as can be checked in Table III,

$$G_P \cdot X_v = X_v \cdot R_u = R_u \cdot T \cdot R_{u'} = R_u \cdot X_{u'} \quad (15)$$

$$G_P \cap X_v = T_P \quad (16)$$

with $\boldsymbol{u} \perp P$. Notice that regular representations are not unique.

When $\boldsymbol{v} \perp P$, $\boldsymbol{u} \parallel v$, $l_{11} = \pm 1$, and G_P is a subgroup of X_v (Table II). Here, the linking displacement is invariant with respect to the x coordinate axis.[2] Consequently, $\boldsymbol{R}_c$ can be expressed as

$$\begin{aligned} \boldsymbol{R}_c &= \textbf{Trans}\ (x''', y''', z''')\ \textbf{Twix}\ (\theta)\boldsymbol{L}'\ \textbf{Twix}\ (\psi) \\ &= \textbf{Trans}\ (x''', y''', z''')\ \textbf{Twix}\ (\theta + l_{11}'\psi)\boldsymbol{L}' \end{aligned} \quad (17)$$

where

$$\boldsymbol{L} = \boldsymbol{L}'\ \textbf{Trans}\ (0, l_{24}, l_{34}).$$

Notice that the necessary and sufficient condition for equality

$$\textbf{Twix}\ (\theta_1)\boldsymbol{L}\ \textbf{Twix}\ (\theta_2) = \textbf{Twix}\ (\psi)\boldsymbol{L} \quad (18)$$

to hold it that $l_{11} = \pm 1$, $l_{24} = 0$, and $l_{34} = 0$. In this case

$$\psi = \theta_1 - l_{11}\theta_2. \quad (19)$$

Example 2. Insertion of a clamp. According to Fig. 4 and imposing that the axes of the cylinders be aligned with the axes of their corresponding holes, the following expressions for both constraints will be obtained:

$$\boldsymbol{R}_{12} = \boldsymbol{A}_{11}\ \textbf{Trans}\ (x_1, 0, 0)\ \textbf{Twix}\ (\theta_1)\boldsymbol{A}_{12}, \quad \boldsymbol{R}_{12}^G = C_u \quad (20)$$

$$\boldsymbol{R}_{23} = \boldsymbol{A}_{21}\ \textbf{Trans}\ (x_2, 0, 0)\ \textbf{Twix}\ (\theta_2)\boldsymbol{A}_{22}, \quad \boldsymbol{R}_{23}^G = C_{u'}. \quad (21)$$

[2] According to the convention relative to the local coordinate frames associated with the geometric elements of definition.

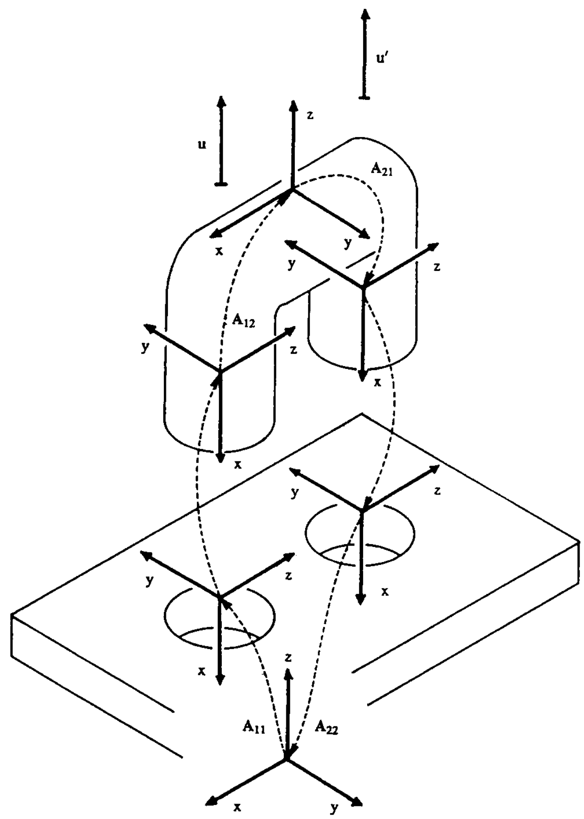

Fig. 4. Insertion of a clamp: g.s.r. and geometric elements of definition.

The composition of both constraints yields

$$\boldsymbol{R}_{12}\boldsymbol{R}_{23} = \boldsymbol{A}_{11}\ \textbf{Trans}\ (x_1, 0, 0)\ \textbf{Twix}\ (\theta_1)\boldsymbol{L} \cdot \textbf{Trans}\ (x_2, 0, 0)\ \textbf{Twix}\ (\theta_2)\boldsymbol{A}_{22} \quad (22)$$

where

$$\boldsymbol{L} = \boldsymbol{A}_{12}\boldsymbol{A}_{21}.$$

Since $\boldsymbol{u}$ and $\boldsymbol{u}'$ are parallel, and according to Table III, $l_{11} = \pm 1$; therefore, the composition of both constraints can be simplified leading to

$$\boldsymbol{R}_{12}\boldsymbol{R}_{23} = \boldsymbol{A}_{11}\ \textbf{Trans}\ (x_1 + l_{11}x_2, 0, 0)\ \textbf{Twix}\ (\theta_1)\boldsymbol{L} \cdot \textbf{Twix}\ (\theta_2)\boldsymbol{A}_{22}. \quad (23)$$

If, in addition to $l_{11} = \pm 1$, $l_{24} = 0$, and $l_{34} = 0$ ($\boldsymbol{u} \bowtie \boldsymbol{u}'$), a further simplification could be carried out and the resulting constraint would be associated with C_u. Actually, C_u and $C_{u'}$ would then stand for the same subgroup.

Expression (23) is a regular representation for the composition of both constraints. It is easy to check that **Trans** $(x_1 + l_{11}x_2, 0, 0)$ **Twix** (θ_1) corresponds to C_u and **Twix** (θ_2) to $R_{u'}$. The simplified term, **Trans** $(x_2, 0, 0)$, which corresponds to $\boldsymbol{R}_{12}^G \cap \boldsymbol{R}_{23}^G = C_u \cap C_{u'}$, encompasses the remaining d.o.f. of body $\mathcal{B}_2$ with respect to $\mathcal{B}_1$, when $\mathcal{B}_3$ is kept rigidly linked, as in this case, to $\mathcal{B}_1$.

This last example seems to be easy to generalize. Let us suppose an open kinematic chain of dimension d relating n bodies, then

$$\dim(\boldsymbol{R}_{1i}) \le d, \ \dim(\boldsymbol{R}_{in}) \le d, \qquad \forall i = 2, \cdots, n-1$$

and

$$d = \dim(\boldsymbol{R}_{1i}) + \dim(\boldsymbol{R}_{in}) - f$$

where f would be the number of d.o.f. of body $\mathcal{B}_i$ with respect to body $\mathcal{B}_1$, if $\mathcal{B}_1$ and $\mathcal{B}_n$ are rigidly linked together. Nevertheless, this reasoning is not strictly true in all cases. It is only true if the number of assignments of variables, resulting from solving the matrix equations (see Section V), equals the dimension of the regular representation.

Although the previous ideas provide a theoretical framework within which it is easy to justify, for instance, when the composition of two constraints can be simplified, they must be complemented with an algorithm to obtain numerical values for the constrained d.o.f. in order to solve problems in practice.

V. Obtaining Numerical Values for the Constrained d.o.f.

Let us consider two bodies, $\mathcal{B}_A$ and $\mathcal{B}_B$, with only one part each (A_1 and B_1, respectively). We are only going to take into account two elements for each of these parts, a_{11}, a_{12}, b_{11}, and b_{12}. With the aim of simplifying the notation, all transformations representing the locations of either parts or elements will be denoted with the same name, of the part or element, in bold capital letters. If two constraints, $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, are imposed and the resulting g.s.r. is the one in Fig. 5(a), from any cycle in the graph, it is possible to obtain a matrix equation as the following one:

$$\boldsymbol{B} = \boldsymbol{A}\boldsymbol{A}_1\boldsymbol{A}_{11}\boldsymbol{R}_1\boldsymbol{B}_{11}^{-1}\boldsymbol{B}_1^{-1} = \boldsymbol{A}\boldsymbol{A}_1\boldsymbol{A}_{12}\boldsymbol{R}_2\boldsymbol{B}_{12}^{-1}\boldsymbol{B}_1^{-1}. \quad (24)$$

Likewise, from the associated GR graph (Fig. 5(b)) one obtains

$$\boldsymbol{A}_{11}\boldsymbol{R}_1\boldsymbol{B}_{11}^{-1}\boldsymbol{B}_{12}\boldsymbol{R}_2^{-1}\boldsymbol{A}_{12}^{-1} = \boldsymbol{A}_{11}\boldsymbol{R}_1\boldsymbol{L}\boldsymbol{R}_2^{-1}\boldsymbol{A}_{12}^{-1} = \boldsymbol{I}. \quad (25)$$

The decomposition of each displacement into its translational and rotational components leads to

$$\mathbf{Trans}\,(\boldsymbol{a}_1)\hat{\boldsymbol{A}}_{11}\,\mathbf{Trans}\,(\boldsymbol{d}_1)\hat{\boldsymbol{R}}_1\,\mathbf{Trans}\,(\boldsymbol{i})\hat{\boldsymbol{L}}\hat{\boldsymbol{R}}_2^t \cdot \mathbf{Trans}\,(-\boldsymbol{d}_2)\hat{\boldsymbol{A}}_{12}^t\,\mathbf{Trans}\,(-\boldsymbol{a}_2) = \boldsymbol{I} \quad (26)$$

hence

$$\mathbf{Trans}\,(\boldsymbol{a}_1 + \hat{\boldsymbol{A}}_{11}\boldsymbol{d}_1 + \hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{R}}_1\boldsymbol{i} - \hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{R}}_1\hat{\boldsymbol{L}}\hat{\boldsymbol{R}}_2^t\boldsymbol{d}_2 - \hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{R}}_1\hat{\boldsymbol{L}}\hat{\boldsymbol{R}}_2^t\hat{\boldsymbol{A}}_{12}^t\boldsymbol{a}_2) \cdot \hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{R}}_1\hat{\boldsymbol{L}}\hat{\boldsymbol{R}}_2^t\hat{\boldsymbol{A}}_{12}^t = \boldsymbol{I}. \quad (27)$$

Equating the rotational and translational components of both sides of (27) yields

$$\hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{R}}_1\hat{\boldsymbol{L}}\hat{\boldsymbol{R}}_2^t\hat{\boldsymbol{A}}_{12}^t = \boldsymbol{I} \quad (28)$$

$$\boldsymbol{a}_1 + \hat{\boldsymbol{A}}_{11}\boldsymbol{d}_1 + \hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{R}}_1\boldsymbol{i} - \hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{R}}_1\hat{\boldsymbol{L}}\hat{\boldsymbol{R}}_2^t\boldsymbol{d}_2 - \hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{R}}_1\hat{\boldsymbol{L}}\hat{\boldsymbol{R}}_2^t\hat{\boldsymbol{A}}_{12}^t\boldsymbol{a}_2 = 0. \quad (29)$$

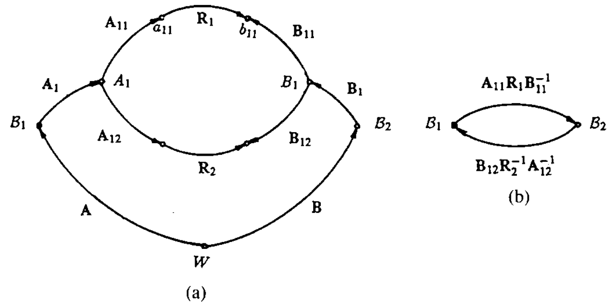

Fig. 5. (a) Example of g.s.r. and (b) its associated GR graph.

Notice that the rotational component can be extracted directly from the original equation, but not the translational component whose expression contains rotational components. The method applied to solve such equations is based on the following fact: first, the rotational component is solved and then also the translational one. This sequencing strategy is the same as that followed in [2]. The procedure derived is powerful enough for solving most assembly problems.

In solving both equations, with the aim of obtaining assignments for the variables (d.o.f.) in constraints $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, one of the following four possible situations will arise:

- *There is no solution.* In Section V-B some sufficient conditions for the identification of these situations will be given.
- *One or several discrete sets of possible values for the d.o.f. are obtained.* In other words, one or more configurations satisfy the constraints (Fig. 6).
- *One or several discrete sets of possible values for some d.o.f. are obtained.* The remaining d.o.f. are linearly related (Fig. 7). A sufficient condition for this situation holds when it is not possible to simplify the rotational component of the regular representation, and this representation does not include more than three **Twix** operators.
- *The d.o.f. do not appear linearly related so that a continuous set of physical configurations satisfying the constraints is obtained.* The study of these situations requires, in general, the use of Mobility Charts [5] that represent the ranges of possible values for each d.o.f. as a function of the values of the others. However, the analysis of these situations is beyond the objectives pursued in solving most common assembly problems. Several examples can be found in [7] (Fig. 8).

Therefore, the problem of finding values for the variables (d.o.f.) associated with constraints can be reduced to the problem of obtaining the cycles appearing in the GR directed graph and solving their corresponding matrix equations. This process allows us to assign values to variables (d.o.f.), to obtain relations between them, or to find inconsistencies.

The algorithm for extracting cycles follows a search procedure that identifies $c - b + 1$ cycles, c being the number of constraints and b the number of bodies.

A set C of cycles in a graph is said to be *complete* if every cycle in the graph can be expressed as a ring sum of cycles in C, and no cycles in C can be expressed as a ring sum of other cycles in C [4]. Thus the matrix equations corresponding to

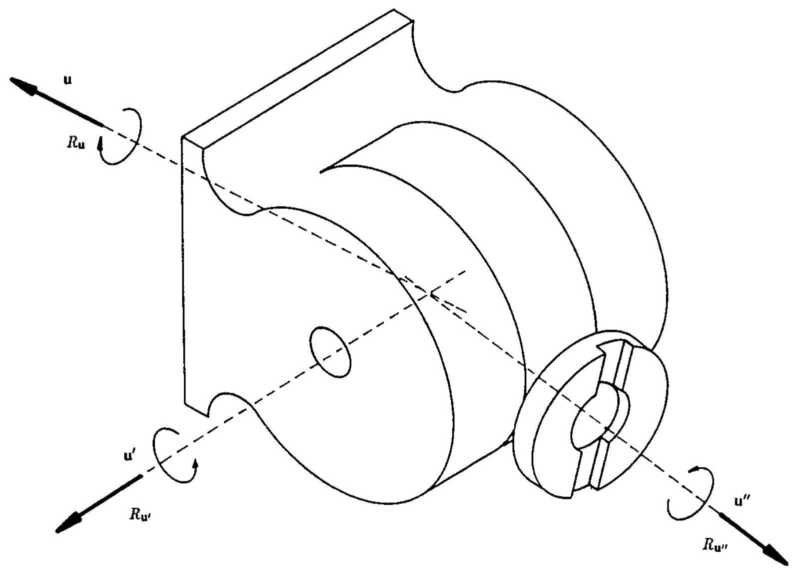

Fig. 6. A Unimate PUMA/560 robot has two discrete possible solutions (flip/noflip configurations) for the three joint angles of its wrist.

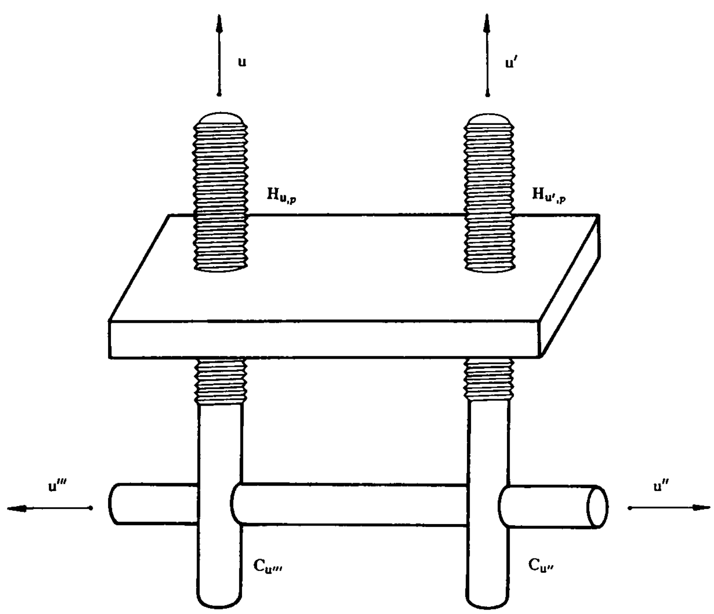

Fig. 7. Assembly with discrete possible values for some d.o.f. The remaining ones are linearly related.

the cycles in a complete set of cycles of a GR graph are independent and no more independent equations can be found. Given an arbitrary connected GR graph, a spanning tree can be easily obtained. Then, when one of the $c - b + 1$ edges in the graph not included in the spanning tree is added to the tree, a unique cycle results. Such cycles are called *fundamental cycles,* and it can be demonstrated that fundamental cycles always constitute a complete set of basic cycles [4]; the converse is not always true.

Unfortunately, there is no theory to guide us in obtaining the simplest set of $c - b + 1$ equations, but in [14] some heuristics can be found.

A. Generated Matrix Equations

The edges of a GR graph are, as stated before, symbolic matrix products premultiplied and postmultiplied by displacements, which are, in general, the product of the transformation from the coordinate frame of a body to one of its parts and from there to one of its elements or the other way around. In order to use the symbolic chains of matrix products in Table I,

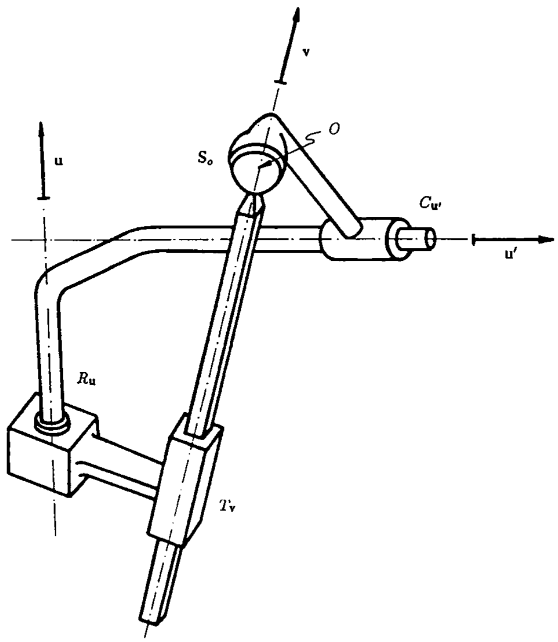

Fig. 8. Assembly with a continuous set of physical configurations satisfying the constraints. The d.o.f. are limited to ranges of possible values.

the local coordinate frames of the geometric elements of definition (vertices, edges, and planes) are chosen according to the convention established in Section III.

The displacement from any coordinate frame to a vertex and its inverse are obvious.

1) Transformation to an Edge: If an edge is represented by a vector $v = ai + bj + ck$ indicating its direction, together with its nearest point (d_1, d_2, d_3) to the origin of the coordinate frame, we have (refer to Fig. 9(a))[3]

$$\alpha = \text{atan2}(c, a)$$

$$\beta = \text{atan2}(b, +\sqrt{a^2+c^2}).$$

The resulting displacement is

$$H_a = \textbf{Trans}\ (d_1, d_2, d_3)\ \textbf{Rot}\ (y, -\alpha)\ \textbf{Rot}\ (z, \beta). \quad (30)$$

2) Transformation to a Plane: If a plane is represented by its unit normal vector (a, b, c) and its distance d to the origin of the coordinate frame of the body to which it belongs, we have (refer to Fig. 9(b))

$$H_p = \textbf{Trans}\ (da, db, dc)\ \textbf{Rot}\ (y, -\alpha)\ \textbf{Rot}\ (z, \beta). \quad (31)$$

B. Resolution of the Rotational Component

After obtaining a matricial equation, its rotational component is extracted and simplified before solving it. This procedure is carried out by applying the following steps:

1) Eliminate all the **Trans** operators.

2) Carry out all possible matrix products.

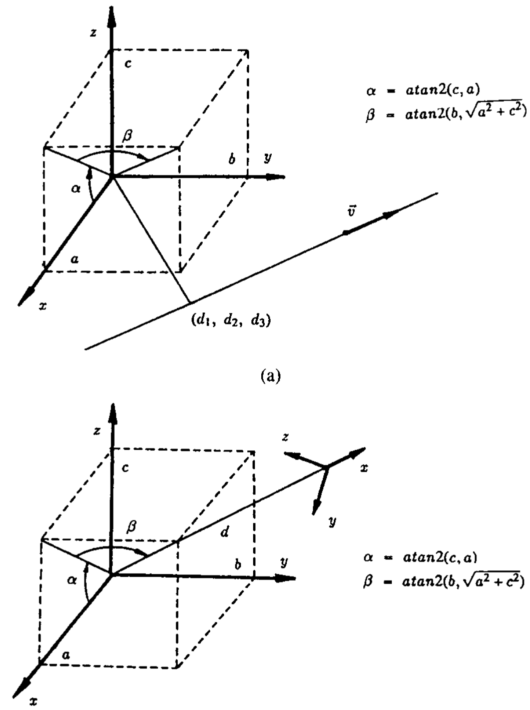

Fig. 9. (a) Transformation to the coordinate frame of an edge. (b) Transformation to the coordinate frame of a plane.

3) Find out the first couple **Twix** (ϕ) A, starting at the rightmost extreme of the chain, where A is a constant matrix. If $\hat{A}$ is x-axis invariant, the vector $(1, 0, 0, 0)^t$ is an eigenvector of this transformation. In such case, the following rewrite rule is applied:[4]

$$\textbf{Twix}\ (\phi)A \Rightarrow A\ \textbf{Twix}\ (a_{11}\phi) \quad (32)$$

where a_{11} is the element (1, 1) of the matrix A. If, after this substitution, two **Twix** operators appear together, the following rule is also applied:

$$\textbf{Twix}\ (\phi_1)\ \textbf{Twix}\ (\phi_2) \Rightarrow \textbf{Twix}\ (\phi_1+\phi_2). \quad (33)$$

4) Go to point 2 if the rules in the preceding step have been applied.

5) The matrix equation at this point is of the form

$$A_1\ \textbf{Twix}\ (l_1)A_2\ \textbf{Twix}\ (l_2) \cdots A_n\ \textbf{Twix}\ (l_n)A_{n+1} = I \quad (34)$$

where A_i are constant matrices and l_i are linear combinations of variables (d.o.f.). If the product $A_{n+1}A_1$ is x-axis invariant, (34) is substituted by

$$A_2\ \textbf{Twix}\ (l_2) \cdots A_n\ A_{n+1}A_1\ \textbf{Twix}\ (a_{11}l_n + l_1) = I \quad (35)$$

where a_{11} belongs to the matrix resulting from the product $A_{n+1}A_1$. Since $A_2 \cdots A_n$ are not x-axis invariant, the process is finished.

[3] The function atan2 in the following equation is used instead of arctan in order to avoid the problem of ambiguity in the angle.

[4] As a convention, the x-axis invariant transformations are propagated backward.

Now, the matrix equation to be solved is

$$\textbf{Twix}\,(l_1)\boldsymbol{B}_1\,\textbf{Twix}\,(l_2)\cdots\boldsymbol{B}_{n-1}\,\textbf{Twix}\,(l_n)=\boldsymbol{B}_n. \quad (36)$$

We will next see that a discrete closed-form solution exists for matrix equations with up to three **Twix** operators. For $n > 3$, a necessary and sufficient condition for (36) to be solvable will be given.

1) Case $n = 1$: The equation to be solved is

$$\textbf{Twix}\,(\theta)=\boldsymbol{A}. \quad (37)$$

A solution exists iff $a_{11} = 1$, and the solution is

$$\theta=\operatorname{atan2}(a_{32},\, a_{22}). \quad (38)$$

2) Case $n = 2$: The equation to be solved is

$$\textbf{Twix}\,(\theta)\boldsymbol{A}\,\textbf{Twix}\,(\phi)=\boldsymbol{B}. \quad (39)$$

A solution exists iff $a_{11} = b_{11}$ and the solution is

$$\phi=\operatorname{atan2}(a_{13}b_{12}-a_{12}b_{13},\, a_{12}b_{12}+a_{13}b_{13}). \quad (40)$$

3) Case $n = 3$: The equation to be solved is

$$\textbf{Twix}\,(\theta)\boldsymbol{A}\,\textbf{Twix}\,(\phi)\boldsymbol{B}\,\textbf{Twix}\,(\psi)=\boldsymbol{C} \quad (41)$$

from which the following equation can be deduced:

$$a\cos\phi+b\sin\phi=c \quad (42)$$

where

$$a=a_{12}b_{21}+a_{13}b_{31}$$
$$b=a_{13}b_{21}-a_{12}b_{31}$$
$$c=c_{11}-a_{11}b_{11}.$$

A solution exists iff

$$(1-a_{11}^2)(1-b_{11}^2)-(c_{11}-a_{11}b_{11})^2\geq 0 \quad (43)$$

and the solutions are

$$\phi_1=\operatorname{atan2}(cb-a\omega,\, ac+b\omega)$$
$$\phi_2=\operatorname{atan2}(cb+a\omega,\, ac-b\omega) \quad (44)$$

where

$$\omega=+\sqrt{(1-a_{11}^2)(1-b_{11}^2)-(c_{11}-a_{11}b_{11})^2}.$$

4) Case $n = 4$: The equation to be solved is

$$\textbf{Twix}\,(\theta)\boldsymbol{A}\,\textbf{Twix}\,(\phi)\boldsymbol{B}\,\textbf{Twix}\,(\psi)\,\boldsymbol{C}\,\textbf{Twix}\,(\gamma)=\boldsymbol{D} \quad (45)$$

from which the following equation can be deduced:

$$a\cos\phi+b\sin\phi+c=d\cos\gamma+e\sin\gamma+f \quad (46)$$

where

$$a=a_{12}b_{21}+a_{13}b_{31}$$
$$b=a_{13}b_{21}-a_{12}b_{31}$$
$$c=a_{11}b_{11}$$
$$d=c_{12}d_{12}+c_{13}d_{13}$$
$$e=c_{13}d_{12}-c_{12}d_{13}$$
$$f=c_{11}d_{11}.$$

Equation (46) has solution iff there exists a δ such that

$$(1-a_{11}^2)+(1-b_{11}^2)-(\delta-a_{11}b_{11})^2\geq 0$$
$$(1-c_{11}^2)+(1-d_{11}^2)-(\delta-c_{11}d_{11})^2\geq 0 \quad (47)$$

and the solutions are

$$\phi_1=\operatorname{atan2}(b|\delta-c|-a\omega_1,\, a|\delta-c|+b\omega_1)$$
$$\phi_2=\operatorname{atan2}(b|\delta-c|+a\omega_1,\, a|\delta-c|-b\omega_1)$$
$$\gamma_1=\operatorname{atan2}(e|\delta-f|-d\omega_2,\, d|\delta-f|+e\omega_2)$$
$$\gamma_2=\operatorname{atan2}(e|\delta-f|+d\omega_2,\, d|\delta-f|-e\omega_2) \quad (48)$$

where

$$\omega_1=+\sqrt{(1-a_{11}^2)+(1-b_{11}^2)-(\delta-a_{11}b_{11})^2}$$
$$\omega_2=+\sqrt{(1-c_{11}^2)+(1-d_{11}^2)-(\delta-c_{11}d_{11})^2}.$$

5) General Case:

Theorem: For $n > 3$, the matrix equation

$$\textbf{Twix}\,(\theta_1)\boldsymbol{A}_1\,\textbf{Twix}\,(\theta_2)\boldsymbol{A}_2\cdots\boldsymbol{A}_{n-1}\,\textbf{Twix}\,(\theta_n)=\boldsymbol{A}_n \quad (49)$$

has solution iff there exist values for $\delta_1, \delta_2, \cdots, \delta_{n-3}$ so that

$$\begin{array}{ll}(1-a_1^2)+(1-a_2^2)-(\delta_1-a_1a_2)^2 & \geq 0\\ (1-a_3^2)+(1-\delta_2^2)-(\delta_1-a_3\delta_2)^2 & \geq 0\\ (1-a_4^2)+(1-\delta_3^2)-(\delta_2-a_4\delta_3)^2 & \geq 0\\ \vdots & \\ (1-a_{n-2}^2)+(1-\delta_{n-3}^2)-(\delta_{n-4}-a_{n-2}\delta_{n-3})^2 & \geq 0\\ (1-a_{n-1}^2)+(1-a_n^2)-(\delta_{n-3}-a_{n-1}a_n)^2 & \geq 0\end{array} \quad (50)$$

where a_i is the element (1, 1) of the matrix A_i.

Proof. It is true for $n = 4$. Let us suppose it is also true for the case $n - 1$ and let us demonstrate the case n.

The matrix equation for the case n can be expressed as

$$\textbf{Twix}\,(\theta_1)\boldsymbol{A}_1\,\textbf{Twix}\,(\theta_2)\boldsymbol{A}_2\cdots\textbf{Twix}\,(\theta_{n-1})= \boldsymbol{A}_n\,\textbf{Twix}\,(-\theta_n)\boldsymbol{A}_{n-1}^t=\Delta_{n-3}. \quad (51)$$

According to the left-hand side of (49), a solution for $\theta_1, \theta_2, \cdots, \theta_{n-1}$ exists iff there are values for $\delta_1, \delta_2, \cdots, \delta_{n-3}$ satisfying

$$\begin{array}{ll}(1-a_1^2)+(1-a_2^2)-(\delta_1-a_1a_2)^2 & \geq 0\\ (1-a_3^2)+(1-\delta_2^2)-(\delta_1-a_3\delta_2)^2 & \geq 0\\ \vdots & \\ (1-a_{n-3}^2)+(1-\delta_{n-4}^2)-(\delta_{n-5}-a_{n-3}\delta_{n-4})^2 & \geq 0\\ (1-a_{n-2}^2)+(1-\delta_{n-3}^2)-(\delta_{n-4}-a_{n-2}\delta_{n-3})^2 & \geq 0\end{array} \quad (52)$$

where δ_{n-3} is the element (1, 1) of the matrix Δ_{n-3}. Therefore, according to the rightmost part of (51)

$$\delta_{n-3}=a_{n-1,11}a_{n,11}+(a_{n-1,12}a_{n,12}+a_{n-1,13}a_{n,13})\cos\theta_n + (a_{n-1,13}a_{n,12}-a_{n-1,12}a_{n,13})\sin\theta_n \quad (53)$$

where $a_{k,ij}$ stands for the element ij of A_k.

Equation (53) has solution for θ_n iff

$$(1-a_{n-1}^2)+(1-a_n^2)-(\delta_{n-3}-a_{n-1}a_n)^2\geq 0. \quad (54)$$

This inequality, together with (52), lead to (50).

□

A very simple effective procedure exists for checking the existence of values for $\delta_1, \delta_2, \cdots, \delta_{n-3}$ in (50). The process starts by finding a range of possible values for δ_1, according to the first inequality. This range is then used for finding a range for δ_2. This process is repeated until the last inequality is reached where, if there exists a value for δ_{n-3} within the corresponding range satisfying the inequality, the matrix equation in (49) has a solution.

If $\delta_{i,l}$ and $\delta_{i,h}$ are the lower and upper bounds of the range for δ_i, respectively, the process for finding the range for δ_{i+1} takes into account that $\delta_{i+1,l}$ and $\delta_{i+1,h}$ have a minimum and a maximum, respectively, at $\delta_i = -a_{i+1}$ and $\delta_i = a_{i+1}$.

C. Resolution of the Translational Component

If it was impossible to solve the rotational component because it contained more than three **Twix** operators, the translational component is not extracted, and the problem is assumed to be weakly constrained. Only the condition in the statement of the theorem is tested.

The extraction of the translational component is carried out starting at the rightmost extreme of the equation and storing, in a vector of three components, the partial results. Let (x, y, x) be the vector of partial results, the following computations are carried out depending on what the next matrix in the chain is

Trans $(a, b, c) : (x+a, y+b, z+c)$

Twix $(\phi) : (x, y\cos\phi - z\sin\phi, y\sin\phi + z\cos\phi)$

$A : A(x, y, z, 1)^t$.

Simultaneously, in order to simplify the expressions, the following rules are applied:

$(+0\,x) \vee (*1\,x) \Rightarrow x$

$(+x\,0) \vee (*x\,1) \Rightarrow x$

$(*\,0\,x) \vee (*x\,0) \Rightarrow 0$

$(*\,z\,(+y\,x)) \Rightarrow (+(*\,z\,y)(*\,z\,x))$

$(+z\,(+y\,x)) \Rightarrow (+z\,y\,x)$

$(*\,z\,(*\,y\,x)) \Rightarrow (*\,z\,y\,x)$.

These rules, as well as the obtained equations, are formulated in LISP style *Cambridge Polish* notation.

In general, several solutions may appear when solving cycles of constraints, as it was shown to happen in solving the rotational component of a chain with three **Twix** operators. A solution is described by means of a GR graph, a list of equations (as sums of products equated to zero), and a table of binded variables (only needed in the intermediate stages of the process).

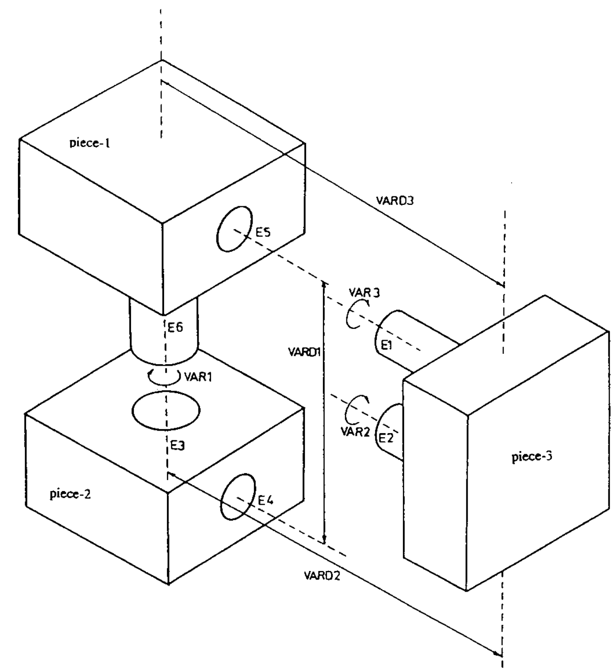

Fig. 10. Simple assembly problem used to show how the implemented system works.

D. Example

Let us consider the simple assembly example in Fig. 10. It deals with the problem of inserting one piece into another and, next, fixing the resulting subassembly by means of a third piece. Let us suppose that the constraints generated by the planner, when reasoning about the compatibilities between the parts of the bodies, are the alignment of the axes (E_2, E_4), (E_1, E_5), and (E_3, E_6). Once the three constraints are inserted into the GR graph, a cycle of length three is extracted. In the corresponding equation, a symbolic operator **Trans** $(x, 0, 0)$ can be simplified. Therefore, one translational d.o.f. will remain in the assembly. In terms of composition of subgroups, the resulting equation can be simplified because $C_u \cap C_{u'}$ is different from the identity displacement—as in example 2 in Section IV-B—since $\boldsymbol{u} \| \boldsymbol{u}'$. Next, the rotational component is extracted and the resulting equation can be also simplified by merging two **Twix** operators, so that an equation with two **Twix** operators is obtained. Afterwards, one angle is directly assigned and the other two are linearly related. The assignments obtained are stored in an associative (*variable-value*) list and the equations relating the d.o.f. not yet assigned are included, as sums of products equated to zero, in a separate list. The original equation is simplified with the obtained assignments and, finally, the three translational equations are extracted and the problem is reduced to four very simple equations, two of which are nonlinear but easy to solve. Fig. 11 shows the output listing of the program.

```
CYCLE LENGTH = 3

((TRANS VARD1 0 0)
 (TWIX VAR1)
 ((-1 0 0 0) (0 1 0 0) (0 0 -1 0) (0 0 0 1))
 ((0 -1 0 0) (1 0 0 0) (0 0 1 0) (0 0 0 1))
 (TRANS VARD3 0 0)
 (TWIX VAR3)
 ((-1 0 0 0) (0 1 0 0) (0 0 -1 0) (0 0 0 1))
 ((1 0 0 0) (0 -4.371139e-8 -1.0 0) (0 1.0 -4.371139e-8 0) (0 0 0 1))
 ((1 0 0 0) (0 1 0 0) (0 0 1 -5) (0 0 0 1))
 ((1 0 0 0) (0 1 0 0) (0 0 1 -5) (0 0 0 1))
 ((1 0 0 0) (0 -4.371139e-8 -1.0 0) (0 1.0 -4.371139e-8 0) (0 0 0 1))
 (TRANS (* -1 VARD2) 0 0)
 (TWIX (* -1 VAR2))
 ((0 -1 0 0) (1 0 0 0) (0 0 1 0) (0 0 0 1)))

PRESIMPLIFIED EQUATION

(((1 0 0 0) (0 1 0 0) (0 0 1 0) (0 0 0 1))
 (TRANS VARD1 0 0)
 (TWIX VAR1)
 ((0 1 0 0) (1 0 0 0) (0 0 -1 0) (0 0 0 1))
 (TRANS VARD3 0 0)
 (TWIX VAR3)
 ((-1.0 0.0 0.0 0.0) (0.0 -1.0 8.742278e-8 10.0)
  (0.0 8.742278e-8 1.0 -4.371139e-7) (0.0 0.0 0.0 1.0))
 (TRANS (* -1 VARD2) 0 0)
 (TWIX (* -1 VAR2))
 ((0 -1 0 0) (1 0 0 0) (0 0 1 0) (0 0 0 1)))

ROTATIONAL COMPONENT

(((1 0 0 0) (0 1 0 0) (0 0 1 0) (0 0 0 1))
 (TWIX VAR1)
 ((0.0 -1.0 8.742278e-8 10.0) (-1.0 0.0 0.0 0.0)
  (0.0 -8.742278e-8 -1.0 4.371139e-7) (0.0 0.0 0.0 1.0))
 (TWIX (+ (* -1 VAR2) (* VAR3 -1)))
 ((0 -1 0 0) (1 0 0 0) (0 0 1 0) (0 0 0 1)))

SIMPLIFIED EQUATION

(((1 0 0 0) (0 1 0 0) (0 0 1 0) (0 0 0 1))
 (TRANS VARD1 0 0)
 ((0 1 0 0) (1.0 0.0 0.0 0.0) (0.0 0.0 -1.0 0.0) (0 0 0 1)) (TRANS VARD3 0 0)
 (TWIX VAR3)
 ((-1.0 0.0 0.0 0.0) (0.0 -1.0 8.742278e-8 10.0)
  (0.0 8.742278e-8 1.0 -4.371139e-7) (0.0 0.0 0.0 1.0))
 (TRANS (* -1 VARD2) 0 0)
 (TWIX (* -1 VAR2))
 ((0 -1 0 0) (1 0 0 0) (0 0 1 0) (0 0 0 1)))

ASSIGNMENTS

 ((VAR1 . 0.0))

 EQUATIONS GENERATED

 (+ (* -10.0 (SIN VAR3)))

 (+ VARD3 (* 1.0 VARD2))

 (+ VARD1 (* 10.0 (COS VAR3)))

 (+ -3.141593 (* -1 VAR2) (* VAR3 -1))
```

Fig. 11. Output listing of the program for the assembly in Fig. 10.

VI. Conclusions

We have presented a procedure for dealing with constraints on the d.o.f. of a set of bodies to be assembled. This procedure is embedded in a general method capable to deal also with two other types of constraints: shape-matching constraints and constraints of nonintersection [16]. This embedding in a broader context imposes certain requirements on the representation of both the objects and the constraints themselves, which in turn give the procedure a wider range of applicability than that characteristic of previous approaches. In this direction, the hierarchical geometric representation used for objects, in terms of *parts* which are further decomposed into its constituent *elements* (planes, edges, and vertices) provides a natural link to interference detection procedures. Furthermore, some spatial relationships—such as *fits* in RAPT or *grasped* in a grasp planning procedure—need not to be explicitly introduced into the system, but can be automatically inferred by an adequate treatment of shape-matching constraints.

A systematization of the possible spatial relationships between bodies (subgroups of the group of spatial displacements), as well as a tabulation of the outcomes of their combination (intersection of subgroups) and chaining (composition of subgroups) is given. The theoretical foundation for this systematization has been taken from the work of Hervé [6].

An algorithmic symbolic procedure, essentially based on that proposed by Ambler and Popplestone [2], but conveniently simplified to increase its efficiency and refined to permit a uniform treatment of some special cases, is explained. Moreover, a theorem that permits eliminating some situations without solution is proved with no commitment in the number of rotational d.o.f.

The procedure to deal with constraints on the d.o.f. described in this paper has other possible applications. For instance, since sensory data often give information which constrains the d.o.f. of objects, the procedure could be used either to determine the positions and orientations of objects or to guide the acquisition of further sensory information to disambiguate between several possibilities.

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