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Analysing Spatial Realizability of Line Drawings Through Edge-Concurrence Tests

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Abstract

This work proves that the realizability of a line drawing without occluding segments can be verified by checking the concurrence of groups of three lines to a single point. These lines are either those supporting segments in the drawing or new ones added during the test itself.

Although this result was essentially already established by W. Whiteley, the presented approach uses the concept of delta-star reductions to obtain all possible spatial realizations and consistent edge-labellings (convex or concave) of a given drawing.

As opposed to well-known algebraic approaches, which require a Waltz filtering preprocessing step before proceeding to the global geometric test, the approach presented herein is based on geometrically interpretable projective conditions which allows an easy localization of the source of eventual inconsistencies.

Keywords: Line Drawing Analysis, Projective Conditions, Spatial Realizability, Spherical Polyhedra.

1 Introduction

A *line drawing* is a drawing containing only *line segments* and *junctions,* points where two or more of these segments meet. A line drawing is said to be *realizable* or *consistent* if it is the orthographic or perspective projection of some three-dimensional scene of polyhedral objects, and *incorrect* or *inconsistent* otherwise. Such a scene is known as a *realization* of the line drawing.

This work addresses the classic problem in Artificial Intelligence and Robotics consisting of deciding whether a drawing is realizable and, if so, giving a parameterization of the space of all its possible realizations.

Since the pioneering work of Guzman [7], the above problem has been the object of research over the past 25 years. In 1971, Huffman [9] and Clowes [1] exploited the fact that there is only a limited amount of different feasible assignments of *concave* (-), *convex* (+) or *occlusive* (>) labels to the segments around a given junction. Once all possible types of junctions are enumerated and registered, everything reduces to a consistent labelling problem with a single constraint: the types of junctions assigned to the end-points of a segment must yield the same label for this segment.

Several other authors refined this scheme to accept a more complex input. For example, in [23] Waltz treated pictures with shadows and cracks. Sanker [16] and Sugihara [19] provided procedures to treat pictures with hidden segments. Drawings of paper-made objects were considered by Kanade in [10].

The main drawback that arises with the labelling scheme is that line drawings having consistent labellings are not guaranteed to be the projection of a polyhedron. For example, the line-drawing in fig. 1a admits the indicated labelling but it does not correspond to any object with planar faces.

One important alternative to the labelling scheme is the use of reciprocal figures in a dual space. Reciprocal figures were already used more than a century ago by Maxwell [12] and Cremona [6] for graphical calculus in mechanics. The idea has been rediscovered and used repeatedly (see e.g. Huffman [9], Mackworth [11] or Draper [8]) as a necessary condition: a labelled line drawing can be classified as inconsistent if it does not admit a reciprocal figure. However, again, only a necessary condition for correctness is obtained.

In 1982, K. Sugihara finally proposes a complete set of constraints that characterizes realizable line drawings [20]. Roughly speaking, his fundamental theorem states that a labelled drawing is correct if and only if a system of linear constraints of the form

$$\begin{aligned} Aw &= 0 \\ Bw &\geq 0 \end{aligned} \qquad (1)$$

has a solution, which can be tested by linear programming techniques. Here, the vector w denotes the unknown parameters of the planar faces of the eventual realization, and A and B are matrices derived from the particular {+,-,>}-labelling, the positions of the junctions and the incidence relations between junctions and regions in the drawing.

In 1984, R. Shapira gave a counterexample proving the incorrectness of the theorem [17] and Sugihara rectified it by changing the definitions of A and B to more accurate ones [21].

Although Sugihara's approach seemed to be only applicable to the special case in which line drawings solely contain trihedral junctions, soon after W. Whiteley proved Sugihara's theorems in the general case [25].

A problem with Sugihara's method is that condition (1) is too strict and slight perturbations of vertex positions can make a line drawing incorrect. In a realistic application, it is impossible to guarantee the exact position of objects in a scene and some uncertainty must be taken into consideration. In order to correct superstrict incorrect pictures, Sugihara proposes to delete from *(1)* those constraints that lead to this *superstrictness* by using the purely combinatorial concept of *position-free incidence* structures [21]. However, in [26] W. Whiteley reports several limitations of this technique.

I. Shimshoni and J. Ponce propose a variation of Sugihara's approach [13]. They define a system similar to *(1)* but, unlike Sugihara, they do not eliminate constraints that lead to a superstrict set of equations, but explicitly introduce uncertainty in these constraints. A *necessary* condition for a line drawing to be the correct projection of a polyhedron is that this system admits a solution, which again can be tested using linear programming.

As far as our problem is concerned, the work of some significant combinatorial geometers is often unnoticed by the robotics community. In this sense, H. Crapo and W. Whiteley have been investigating the connection between the realizability of linear scenes and the rigidity of planar bar frameworks ([2], [3], [4], [24], [25] and [27]). They have proved that a line drawing is consistent if and only if the associated planar bar framework supports a non-null pattern of stresses on the bars. Hence, the realizability of a drawing and the rigidity of a planar framework have been proved to be equivalent problems.

This paper is structured as follows. Section 2 introduces the used notation and defines the type of drawings we treat. In section 3 several well-known projective conditions are reviewed and it is shown that they all can be subsumed by applying a unique test of concurrence on groups of three lines. Section 4 introduces the idea of delta-star reductions, a set of operations used in section 5 to illustrate a novel consistency test, which is fully formalized in section 6. Based on this result, an algorithm to obtain labelled realizations of a given drawing is presented in section 7. Conclusions and points that deserve further attention are finally discussed in section 8.

2 Notation and hypotheses

We assume that every junction in a line drawing is common to at least two line segments and that the segments partition the plane of the drawing into several polygonal *regions*.

The *incidence structure* of a line drawing L is a planar and connected graph $G(L) = (J, S)$ where J is the set of junctions of L and S is the set of line segments. There is a one-to-one correspondence between the elements in a consistent line drawing and the elements of the polyhedral scenes it represents: junctions correspond to *vertices*, line segments to *edges*, and polygonal regions to *faces*.

Although it induces an abuse of language, we will refer to the *line of support* of a given edge l or the *plane of support* of a given face φ by using the terms *line* l and *plane* φ, respectively.

The line drawings are supposed to contain no *occlusive* line segments. That is, every segment represents the intersection of two *adjacent* faces in 3-space. This even holds for the segments of the outer contour of the drawing which, hence, represent coplanar edges in 3-space.

Under these assumptions, the realization of a correct drawing is a spherical polyhedron and, therefore, the class of drawings considered is restricted to those whose incidence structure is planar, edge 3-connected and vertex 2-connected [3].

As briefly described in the conclusions, it is possible to relax the non-occlusivity condition, but this assumption greatly simplifies the treatment given below.

3 Edge-concurrence subsumes all projective conditions

There are several well-known projective conditions that a line drawing should accomplish in order to correctly represent the projection of a polyhedral scene (see e.g. Crapo

[4], [5], Whiteley [24], Sabater [15], Sugihara [20]). We now review some of them.

The *edge alignment condition* (fig. 1a, left) states that if two different edges of a polyhedron share the same two faces, then the two edges must be aligned.

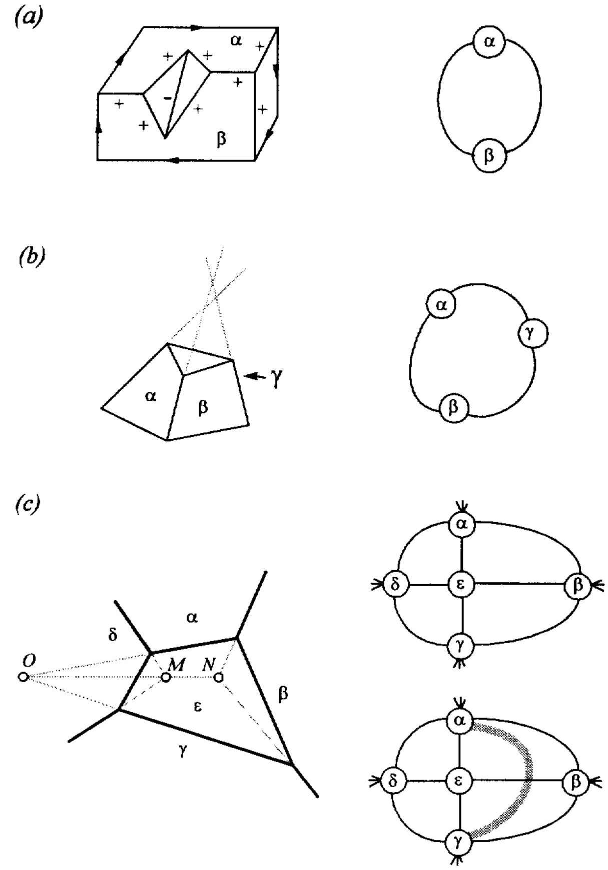

Fig. 1. *Edge-alignment (a), edge-concurrence (b) and n-calotte (c) conditions.*

The *edge concurrence condition* (fig. 1b, left) says that given three faces of a polyhedron, if any two of them share a common edge, then these edges must all be concurrent to the same point. This is true even in the case that the edges are parallel, since we can view them as embedded in projective 3-space, allowing the existence of "points at infinity" [18].

In [4] and [5] Crapo refers to the *n-calotte condition* which imposes constraints on the concurrence of all edges incident to a given *n*-gonal face of a polyhedron. We describe here the case $n = 4$. Given a quadrilateral face of a polytope such as face ε in fig. 1c, left, consider the point *M* of intersection of line $\delta - \alpha$ with line $\delta - \gamma$, and point *N*, where $\alpha - \beta$ intersects with $\gamma - \beta$. Then, *M* and *N* must be aligned with *O*, where $\alpha - \varepsilon$ intersects with $\varepsilon - \gamma$, because *M*, *N* and *O* must lie on the intersection of planes α and γ.

The above three conditions involve two types of geometric elements of the line drawing: regions and line segments. Then, it seems natural to express them in terms of the topology of what we call the *constraint graph*, a graph $G_r(L) = (V, E)$ which contains a vertex in V for each region in the line drawing, and an edge in E for each line segment separating two adjacent regions.

Now, to test the edge-alignment condition, we just gather all pairs of vertices $v \in V$ with parallel edges between them and judge whether the coefficients of their corresponding segments are equal or not (fig. 1a, right).

In order to test the edge-concurrence condition, we detect all cycles of length three in $G_r(L)$ and judge whether the corresponding segments meet at a single point.

The 4-calotte condition in fig. 1c clearly reduces to the three edge-concurrence tests for the sets of lines

$$\{\delta - \alpha, \delta - \gamma, MN\},$$
$$\{\alpha - \beta, \gamma - \beta, MN\},$$
$$\{\alpha - \varepsilon, \gamma - \varepsilon, MN\}.$$

This cannot be directly expressed in terms of the topology of $G_r(L)$ (fig. 1c, top-right) because the drawing does not provide the projection of line *MN*. However, by simply adding a *fictitious edge* to $G_r(L)$ corresponding to the unknown projection of line $\alpha - \gamma$ (fig. 1c, bottom-right), all relevant 3-cycles emerge. Edge-concurrence of the 3-cycles $\alpha, \delta, \gamma, \alpha$, and $\alpha, \beta, \gamma, \alpha$ constrains fictitious line $\alpha - \gamma$ to meet *M* and *N*, respectively. Once the location of $\alpha - \gamma$ is known, the 4-calotte condition is finally verified when checking the 3-cycle $\alpha, \varepsilon, \gamma, \alpha$.

Note that, in general, once a fictitious edge is fixed, this information can be *propagated* and used to fix other fictitious edges.

We say that a constraint graph is *globally consistent* if we can find values for its fictitious edges in such a way that all edge-concurrence conditions corresponding to all 3-cycles in the graph are satisfied.

An important question arises: Is it possible that the realizability of a line drawing can be decided by solely testing edge-concurrence conditions on 3-cycles of either known or fictitious edges? W. Whiteley provided an affirmative answer to this question in 1991 [27].

Let $G_r(L)$ be the constraint graph of a line drawing. Then, arbitrarily choose one face φ of the polyhedron represented by L and construct a new constraint graph, $G_r^e(L)$, by extending $G_r(L)$ with all fictitious edges of the form (φ, x) that represent the intersection of φ with any other face x of the polyhedron. Although explained in a different language in [27], Whiteley's theorem states as follows.

Theorem. *(Whiteley) L is realizable if and only if* $G^e_r(L)$ *is globally consistent.*

This theorem not only unifies all projective conditions to a single one, edge-concurrence, but also proves that not all possible fictitious edges have to be added to $G_r(L)$ to obtain a sufficient set of projective conditions.

4 Delta/star reductions

It is always possible to reduce any spherical polyhedron to a simplex by applying a finite sequence of the two following operations:

- The simplicial completion (fig. 2a), which eliminates a triangular face by extending its three neighboring faces as far as their common point of intersection.
- The simplicial elimination (fig. 2b), which simply removes a trihedral vertex by cutting it through the plane defined by its three neigboring vertices, obtaining a new triangular face.

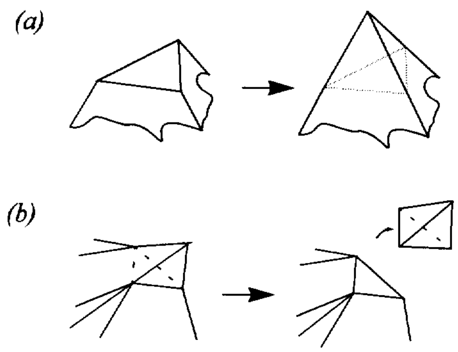

Fig. 2. *Simplicial completion (a) and elimination (b).*

We do not prove this assertion here but let us just mention that the application of at least one of these two operations is guaranteed by the fact that any spherical polyhedron has either a vertex of degree three or a triangular face [14].

We can reason in a similar way when checking the realizability of a line drawing. If a line drawing is consistent, then it has to be possible to transform it to the projection of a simplex by means of some reduction steps analogous to the simplicial operations in the 3D case. This leads us to the definition of the delta/star reductions (Δ / Y reductions, for short).

The projection of the simplicial completion operation onto a plane induces the four different types of *delta-to-star* ($\Delta \rightarrow Y$, for short) reductions shown in fig. 3a. Each of these operations adds a new junction which corresponds to the new trihedral vertex appearing in the polyhedron. Those segments that meet at a junction of degree 3 in the original triangle will be called *simple segments.*

The projection of the simplicial elimination operation leads to four types of *star-to-delta* ($Y \rightarrow \Delta$, for short) reductions (fig. 3b). A new triangular region appears

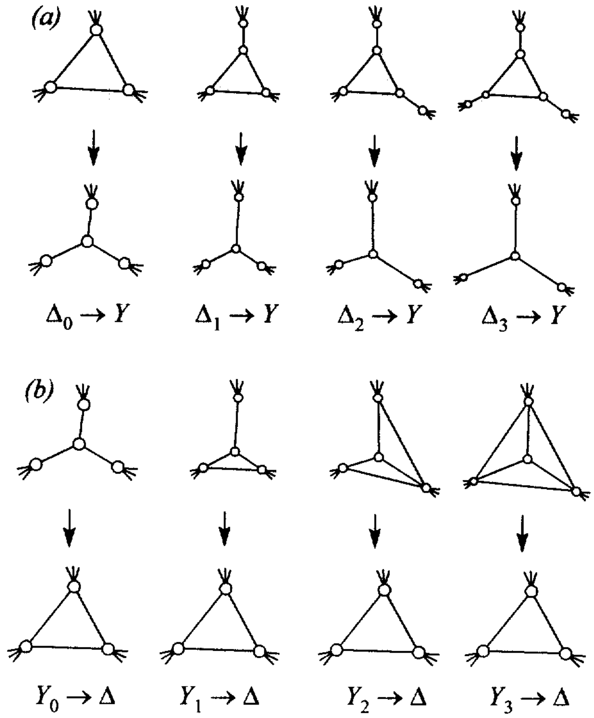

Fig. 3. $\Delta \rightarrow Y$ *(a) and* $Y \rightarrow \Delta$ *(b) operations.*

within the three original junctions of the star.

It is always possible to reduce a correct line drawing of a polyhedron to the projection of a simplex by applying these operations (for details see [14], which essentially follows [22]). How this can be used to test the consistency of a drawing will soon be clarified in the next section by means of an example, and formalized as a theorem in section 6, but first it is worth noting that:

- It can be easily seen that $Y_1 \rightarrow \Delta$, $Y_2 \rightarrow \Delta$, and $Y_3 \rightarrow \Delta$ can be expressed as combinations of $\Delta_1 \rightarrow Y$ and $Y_0 \rightarrow \Delta$ (fig. 4). This fact allows us to reduce the number of strictly necessary reductions.
- In order to apply a $\Delta \rightarrow Y$ reduction, the position of the new junction must be known. This position is clear in $\Delta_2 \rightarrow Y$ and $\Delta_3 \rightarrow Y$ because it is totally constrained to lie in the point where the simple segments meet. In $\Delta_1 \rightarrow Y$ this position is undetermined and must lie on the line of support of the unique simple segment available. In $\Delta_0 \rightarrow Y$ the position is totally free.

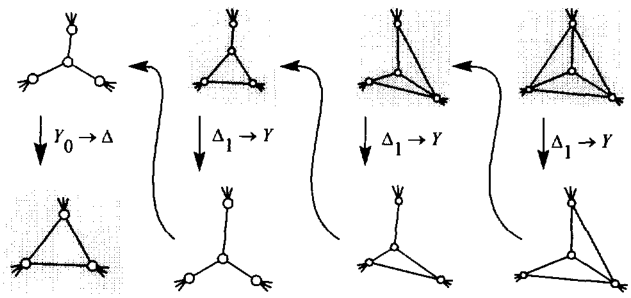

Fig. 4. *$Y_1 \to \Delta$, $Y_2 \to \Delta$, and $Y_3 \to \Delta$ expressed as combinations of $\Delta_1 \to Y$ and $Y_0 \to \Delta$.*

- In a $\Delta \to Y$ reduction, the three regions neighboring the edges of the triangle (delta) correspond to three faces in 3-space for which the intersection of any two of them is sometimes provided in the drawing by simple segments. When this is not the case, we can still make use of these intersection lines adding them to the corresponding constraint graph as fictitious edges, in the same way as we did in the 4-calotte example of fig. 1c. Despite the lack of information on the position of the new junction, the use of the same propagation mechanism can yield to the complete determination of its location. As a consequence, in $\Delta_1 \to Y$ and $\Delta_0 \to Y$ geometric propagation through fictitious edges may fix the position of the new junction.
- Each $\Delta \to Y$ reduction induces a local geometric consistency test. Indeed, the new junction represents the point of intersection of the three faces around the triangle. Hence, in $\Delta_3 \to Y$ the three simple segments must be concurrent. In the rest of $\Delta \to Y$ operations the concurrence test involves simple and fictitious segments. The test is then delayed until all of them become determined by geometric propagation after other reduction steps.
- $Y \to \Delta$ reductions are represented in a constrains graph by adding a vertex (corresponding to the new triangular face) and three edges. Moreover, before applying them, one must be sure that the position of the new vertex of the 3-star is consistent with the rest of the drawing. This can only be done when the rest of the drawing is known to be consistent, and hence this reduction must be delayed until then.

5 Applying Δ/Y reductions to an example

As we have already seen, Δ/Y reductions may add new fictitious edges and vertices to the constraint graph, thus creating new 3-cycles and inducing new edge-concurrence tests. The goal is to prove that the overall consistency of the drawing can be checked by simply verifying all concurrence tests implicit in the constraint graph, once this has been extended with all fictitious edges and faces corresponding to Δ/Y reductions. Before proving this, we illustrate this procedure with an example.

The drawing in fig. 5a can be easily reduced to the projection of a simplex by means of four $\Delta \to Y$ reductions, one for each of the four triangles in it. Let us suppose that

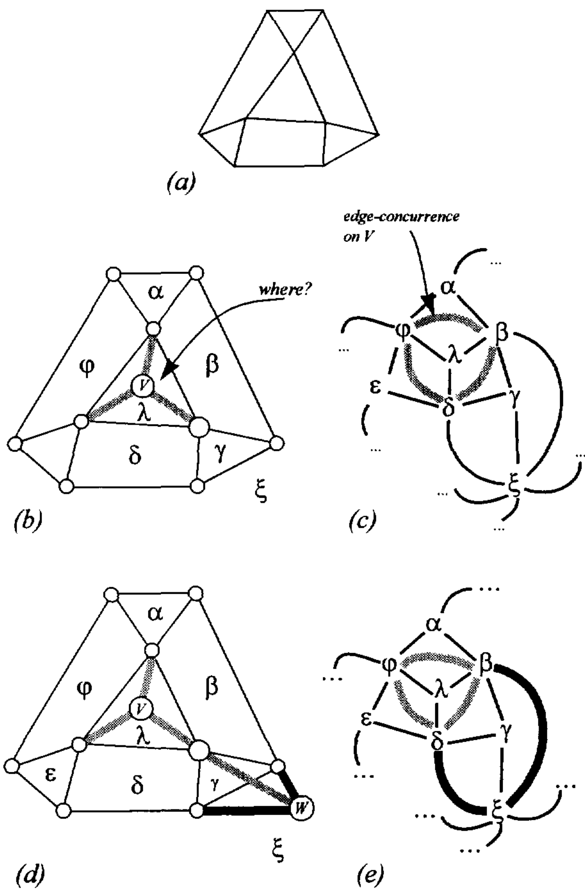

Fig. 5. *Checking the consistency of a truncated pyramid.*

the first reduction is applied over the central triangle, which has no simple edges. This is a $\Delta_0 \to Y$ reduction and, for the moment, it is impossible to tell where the new central junction V should lie to keep the consistence of the drawing (fig. 5b). In the corresponding constraint graph this is translated into the addition of a new 3-cycle with three fictitious edges (the thick grey ones in fig. 5c).

We go on by applying a $\Delta_2 \to Y$ reduction to the lower-right triangle (fig. 5d). This time, the position of the new junction is completely determined. Actually, we have three edges that must be concurrent, namely (β, ξ), (ξ, δ) and (δ, β). Since from the line drawing we know the location of (β, ξ) and (ξ, δ), we are able to fix edge (δ, β). This corresponds to the verification of the edge-concurrence condition on the 3-cycle $\delta, \xi, \beta, \delta$. This allows us to "propagate" this new information to fix edge (δ, β) in the first 3-cycle $\varphi, \delta, \beta, \varphi$.

We now proceed analogously with another $\Delta \rightarrow Y$ operation over the lower-left triangle. Again, the edge-concurrence condition on the 3-cycle $\xi, \varphi, \delta, \xi$ fixes the location of edge (φ, δ), which permits to fix the position of vertex V. Finally, a $\Delta \rightarrow Y$ operation on the upper triangle leads to the unique consistency test of the whole process: if the drawing is consistent, edge (φ, β) must be concurrent with the two previously determined edges (φ, δ) and (δ, β).

6 A novel consistency test

Consider a line drawing L together with its constraint graph $G_r(L)$ and a sequence s of Δ / Y reductions taking L to the projection of a simplex. We build an *extended constraint graph,* $G_r^*(L)$, by adding to $G_r(L)$ all fictitious edges between non-adjacent vertices of $G_r(L)$, all fictitious edges corresponding to all $\Delta \rightarrow Y$ reductions in s and all fictitious faces and edges corresponding to $Y \rightarrow \Delta$ reductions in s.

Theorem. *L is the drawing of a spherical polyhedron if and only if $G_r^*(L)$ is globally consistent.*

Proof. ($\Rightarrow$) If L represents a spherical polyhedron all groups of three edges corresponding to intersections of three pairwise adjacent faces must be concurrent and $G_r^*(L)$ is globally consistent.

($\Leftarrow$) Suppose that $G_r^*(L)$ is globally consistent. We prove that L is realizable by induction on the number n of reduction steps in the sequence s.

For $n = 1$ we distinguish two cases. If the reduction is a $\Delta \rightarrow Y$, then L must be combinatorially equivalent to one of the following planar, vertex 2-connected and edge 3-connected line drawings:

The first (leftmost) drawing is always realizable regardless of the position of its junctions. Obviously, no 3-cycle is inconsistent on its graph $G_r^*(L)$ and the theorem holds. The second drawing is realizable provided that the three simple segments incident to the vertices of the bold triangle are all concurrent, which is guaranteed by the consistency of $G_r^*(L)$ and the theorem again holds.

If the reduction is a $Y \rightarrow \Delta$ then the possible line drawings are combinatorially equivalent to one of these two triangulated realizable drawings:

Now, suppose the theorem is valid for $n = k$. We prove that the validity extends to the case $n = k+1$. For this, we distinguish whether the first reduction step applied to L to obtain a new drawing L' is a $\Delta \rightarrow Y$ or a $Y \rightarrow \Delta$. For the first case fig. 6 depicts the relevant segments of L

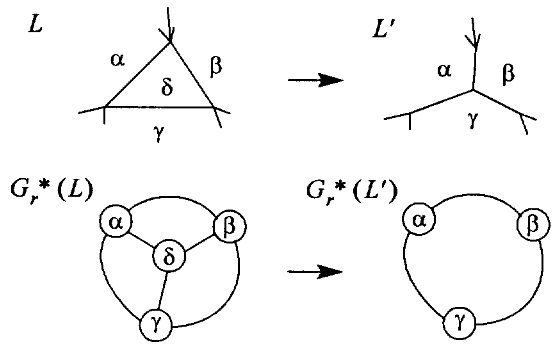

Fig. 6. *Proof of the theorem for $n=k+1$: $\Delta \rightarrow Y$ case.*

and L', and the edges of their constraint graphs.

If $G_r^*(L)$ is consistent, any of its subgraphs is also consistent, in particular so is $G_r^*(L')$. Since L' needs k steps to be reduced to the projection of a simplex, by induction hypothesis L' is realizable. If L' is realizable, then so is L, since a realization of L is simply obtained by cutting the polyhedron corresponding to L' by a plane intersecting the three vertices of the 3-star $\alpha - \beta - \gamma$.

When the first reduction is a $Y \rightarrow \Delta$ (fig. 7), recall that we only have to consider the case $Y_0 \rightarrow \Delta$.

The only difference between $G_r^*(L)$ and $G_r^*(L')$ are edges (α, β), (β, γ) and (γ, α), fictitious in $G_r^*(L')$ but known in $G_r^*(L)$. Then, if $G_r^*(L)$ is consistent, so is $G_r^*(L')$ since we can fix these three fictitious edges to the values they have in the three corresponding edges in $G_r^*(L)$. Now, by induction hypothesis, L' has a realization R' and a realization R of L can be easily obtained by extending the three faces α, β

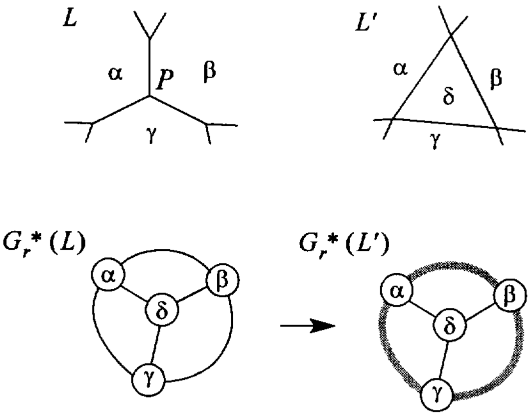

Fig. 7. *Proof of the theorem for $n=k+1$: $Y \rightarrow \Delta$ case.*

and γ of R' as far as their common point of intersection, which must project onto P because edges (α, β), (β, γ) and (γ, α) were fixed as they are in $G_r^*(L)$. □

7 Labelling edges

Once a drawing is known to be consistent, the proposed consistency test also provides a way to obtain its realizations. To this end, just take the final projection of the simplex, give arbitrary heights to its four vertices, and apply the spatial operations corresponding to the inverses of the Δ/Y reductions in s. It is not difficult to show that the degrees of freedom of these realizations only appear when a $Y_3 \rightarrow \Delta$ reduction is "undone" and an arbitrary height for the vertex in the new 3-star (of type Y_3) must be chosen [14]. Moreover, the concave (-) or convex (+) shape of an edge directly depends on the value taken for this height.
As an example, fig. 8 shows the evolution of a drawing (fig. 8a) and its spatial realization (fig. 8b) after undoing two reductions, a $Y_3 \rightarrow \Delta$ and a $Y_2 \rightarrow \Delta$. If the initial state of the polyhedron is that in fig. 8b, we clearly see that, once a choice is made for the height of vertex V, the edges arround it take a concave or convex shape (fig. 8c and d, respectively) which, in turn, determine the shape of subsequent new edges around W.

8 Conclusions

We have presented a novel approach to solve the problem of deciding whether a drawing corresponds to the projection of a polyhedron. When compared to former approaches, it exhibits two main advantages. First, the usual problem of superstrictness arising in algebraic approaches can be overcome by simply allowing some error tolerance in all concurrence tests. Additionally, when one of these tests fails, involved vertices can be easily identified and the source of inconsistency, located. Second, consistent edge-labellings are synthesized as a result of the reconstruction, instead of being a required input.

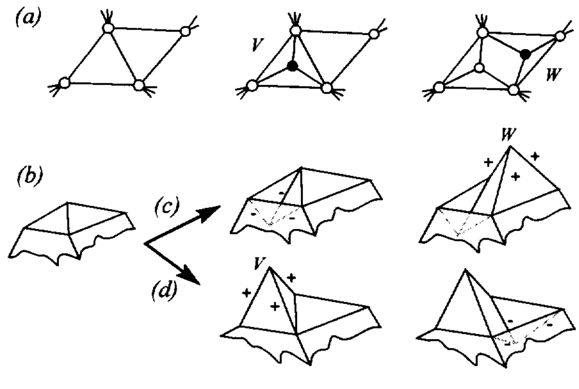

Fig. 8. *Convexity and concavity emerges*

Although our approach is currently limited to drawings without occlusive segments, it clearly establishes the basis for future developments. In this sense, if a drawing contains occlusive segments, then our test will probably judge it as inconsistent. Nevertheless, the drawing can still be correctly interpreted if we mark some of its segments as occlusive and properly add new edges to its incidence structure. For example, the (partial) drawing in fig. 9a is inconsistent, but if we consider that segments l and m are occlusive and add an extra segment n to it, then our procedure can recover the three dimensional shape as shown in fig. 9c. This is part of our current research.

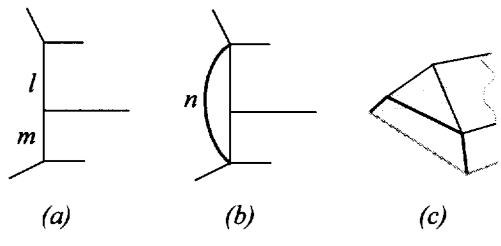

Fig. 9. *Treating occlusive segments*

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