On the Reconstruction of an Image from its Moments

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Abstract— **An image can be seen as an element of a vector space so that it can be expressed in terms of a series expansion of any non necessarily orthogonal base of this space. This paper shows how a matrix-based formulation of this fact permits deriving a new reconstruction method of an image from its geometric moments where the basis functions used in the reconstruction and those used to obtain the moments do not necessarily define the same subspace. This permits introducing constraints relative to the bandwidth or the spatial resolution of the image to be reconstructed. Moreover, it is shown that, by exploiting the algebraic properties of the involved matrices as well as the properties of computer arithmetic, accurate solutions to this problem in spite of its ill-conditioning can be obtained.**

I. Introduction

The reconstruction of an image from a set of its moments is not necessarily unique. In other words, it is an ill-posed problem. Therefore, all possible methods to solve it must impose extra constraints so that the solution becomes unique. It can be shown that the standard least-squares reconstruction method solves this ill-posedness by assuming constraints on the unknown moments themselves. On the contrary, the reconstruction method proposed in this paper permits introducing constraints that can be interpreted in terms of image properties, such as its bandwidth or spatial resolution.

The standard reconstruction method of an image from some of its moments is based on the least-squares approximation of the image using orthogonal polynomials [12], [10], [8]. Polynomials are the most intuitive choice among all possible orthogonal basis functions because they can be easily related to the monomial functions that are used to obtain geometric moments. Legendre and Zernike polynomials were first used in [12]. They are orthogonal polynomials for continuous variables in rectangular and polar coordinates, respectively. However, they are not orthogonal for discrete variables, contrary to what is assumed by some authors [10], [4]. Tchebichef polynomials were used in [5] and [8] which are orthogonal polynomials in arbitrary discrete domains. Independently of the chosen set of polynomials, the standard method assumes null projection coefficients onto the chosen polynomial set of order higher than the maximum order of available moments. This solves the ill-posedness and the solution becomes unique. In order to avoid this assumption, which is difficult to interpret in terms of the image properties, a maximum entropy method was proposed in [9]. It consists in obtaining the image with maximum entropy with the desired moments. Solving the problem using Lagrange multipliers permits to obtain an explicit form of the reconstructed image in terms of an exponential function. Alternatively, [7] proposes minimizing the divergence of the image, instead of maximizing its entropy, using also a variational approach. Unfortunately, both approaches assume a continuous domain for the image.

This paper generalizes the standard method so that the least-squares approximation using orthogonal polynomials can just be seen as a particular case of the general technique presented here.

This paper is structured as follows. The next section introduces the necessary mathematical background. Section III reformulates the standard method in terms of the presented formalism. Section IV generalizes the result to other orthogonal basis different from polynomials. Section V deals with the numerical conditioning of the problem and, finally, section VI contains the conclusions.

II. A matrix-based image series approximation

Following the same matrix notation introduced in [6], let $\mathbf{Z}_{mn}$ denote a matrix of size $m \times n$ and $\mathbf{Z}_{mn}[k,l]$, its element (k,l), where $1 \le k \le m$ and $1 \le l \le n$. Superscripts are also used to denote any parameter on which a matrix depends. Two unary matrix operations are used: $(\cdot)^t$ denotes the transpose of a given matrix; and $(\cdot)^{-1}$, its inverse. To avoid confusions, matrices are always embraced by parenthesis when superscripts refer to power or transpose.

Any discrete image of size $a \times b$, $\mathbf{I}_{ab}$, can be seen as a vector in $\Re^{a\times b}$ or, alternatively, as a bidimensional function that maps all the points of the uniform lattice $\{1,2,\ldots,a\}\times\{1,2,\ldots,b\}$ onto real values. Then, $\mathbf{I}_{ab} \in \Re^{a\times b}$ can be uniquely expressed as a linear combination of the functions of a basis set, i.e., a set containing ab linearly independent bidimensional functions, which will be denoted by $\{\mathbf{\Xi}^{k\,l}_{ab}\}$, $k = 0,\ldots,a-1$ and $l =$

$0,\ldots,b-1$. In other words, $\mathbf{I}_{ab} = \sum_{k=0}^{a-1}\sum_{l=0}^{b-1} \alpha_{k\,l}\, \mathbf{\Xi}_{ab}^{k\,l}$.

Definition 1 (Basis matrix) The functions in any basis set are assumed to be separable and equally defined for both coordinates, i.e., $\mathbf{\Xi}_{ab}^{k\,l} = \boldsymbol{\phi}_a^k\ (\boldsymbol{\phi}_b^l)^t$, where $\boldsymbol{\phi}_a^k$ and $\boldsymbol{\phi}_b^l$ are vectors which will be grouped in matrices of the form $\mathbf{\Phi}_{pq} = \begin{pmatrix}\boldsymbol{\phi}_p^0 & \ldots & \boldsymbol{\phi}_p^{(q-1)}\end{pmatrix}$, called *basis matrices.*

Definition 2 (Gram matrix) The matrix $\mathbf{\Gamma}_q^p = (\mathbf{\Phi}_{pq})^t\ \mathbf{\Phi}_{pq}$, containing the inner products between the elements of the corresponding basis matrix, is called a *Gram matrix.*

Note that, since $\mathbf{\Gamma}_q^p[k+1,l+1] = <\boldsymbol{\phi}_p^k, \boldsymbol{\phi}_p^l>$, the Gram matrices are diagonal for orthogonal basis sets and the identity for orthonormalized basis.

Definition 3 (Projection matrix) The matrix containing the projection coefficients of image $\mathbf{I}_{ab}$ onto the first $m \times n$ elements of $\{\mathbf{\Xi}_{ab}^{k\,l}\}$ are called *projection matrices*, which can be expressed as $\mathbf{\Omega}_{mn} = (\mathbf{\Phi}_{am})^t\ \mathbf{I}_{ab}\ \mathbf{\Phi}_{bn}$.

Note that $\mathbf{\Omega}_{mn}[k+1,l+1] = <\mathbf{I}_{ab}, \mathbf{\Xi}_{ab}^{kl}> = (\boldsymbol{\phi}_a^k)^t\ \mathbf{I}_{ab}\ \boldsymbol{\phi}_b^l$.

Definition 4 (Expansion matrix) The image $\mathbf{I}_{ab}$ can be partially expanded in terms of the first $m \times n$ elements of $\{\mathbf{\Xi}_{ab}^{k\,l}\}$ as

$$\hat{\mathbf{I}}_{ab}^{mn} = \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} \lambda_{kl}\, \mathbf{\Xi}_{ab}^{kl} = \mathbf{\Phi}_{am}\ \mathbf{\Lambda}_{mn}\ (\mathbf{\Phi}_{bn})^t,$$

where $m \le a$, $n \le b$, and $\mathbf{\Lambda}_{mn}[k{+}1,l{+}1] = \lambda_{kl}$. If λ_{kl} is chosen so that the truncation error is minimized using the least-squares error criterion, $\mathbf{\Lambda}_{mn}$ denotes what it is called an *expansion matrix.*

Lemma 1: The series approximation of an image, in the least-squares sense, can be expressed as

$$\begin{aligned}\hat{\mathbf{I}}_{ab}^{mn} &= \mathbf{\Phi}_{am}\ \mathbf{\Lambda}_{mn}\ (\mathbf{\Phi}_{bn})^t \\ &= \mathbf{\Phi}_{am}\ (\mathbf{\Gamma}_m^a)^{-1}\ \mathbf{\Omega}_{mn}\ (\mathbf{\Gamma}_n^b)^{-1}\ (\mathbf{\Phi}_{bn})^t \\ &= \mathbf{\Phi}_{am}\ ((\mathbf{\Phi}_{am})^t\ \mathbf{\Phi}_{am})^{-1} \mathbf{\Omega}_{mn}\ ((\mathbf{\Phi}_{bn})^t\ \mathbf{\Phi}_{bn})^{-1}\ (\mathbf{\Phi}_{bn})^t \\ &= (\mathbf{\Phi}_{am})^-\ \mathbf{\Omega}_{mn}\ (\mathbf{\Phi}_{bn})^+, \qquad (1)\end{aligned}$$

where $(\cdot)^-$ and $(\cdot)^+$ stand for the left and right Moore-Penrose pseudoinverses.

Proof: Given $\mathbf{I}_{ab}$ and the first $m \times n$ elements of $\{\mathbf{\Xi}_{ab}^{kl}\}$, it can be easily shown that the corresponding expansion matrix can be expressed in terms of the projection and Gram matrices as $\mathbf{\Lambda}_{mn} = (\mathbf{\Gamma}_m^a)^{-1}\ \mathbf{\Omega}_{mn}\ (\mathbf{\Gamma}_n^b)^{-1}$, where $m \le a$ and $n \le b$. The lemma follows directly. ■

Corollary 1: If the basis set $\{\overline{\mathbf{\Xi}}_{ab}^{kl}\}$ is orthonormal — we use an overline to distinguish it from the general case — the least-squares approximation of the image can be expressed as $\overline{\mathbf{I}}_{ab}^{mn} = \overline{\mathbf{\Phi}}_{am}\ \overline{\mathbf{\Omega}}_{mn}\ (\overline{\mathbf{\Phi}}_{bn})^t$ because $\overline{\mathbf{\Omega}}_{mn} = \overline{\mathbf{\Lambda}}_{mn}$.

Lemma 2: Given the projection matrix $\mathbf{\Omega}_{mn} = (\mathbf{\Phi}_{am})^t\ \mathbf{I}_{ab}\ \mathbf{\Phi}_{bn}$, the series approximation of $\mathbf{I}_{ab}$ in terms of an arbitrary orthonormal basis $\overline{\mathbf{\Phi}}_{am}$ can be expressed as:

$$\begin{aligned}\overline{\mathbf{I}}_{ab}^{mn} &= \overline{\mathbf{\Phi}}_{am}\ ((\mathbf{\Phi}_{am})^t\ \overline{\mathbf{\Phi}}_{am})^{-1}\ \mathbf{\Omega}_{mn}\ ((\overline{\mathbf{\Phi}}_{bn})^t\ \mathbf{\Phi}_{bn})^{-1}(\overline{\mathbf{\Phi}}_{bn})^t \\ &= \overline{\mathbf{\Phi}}_{am}\ (\mathbf{C}_m^a)^{-1}\ \mathbf{\Omega}_{mn}\ \left((\mathbf{C}_n^b)^t\right)^{-1}\ (\overline{\mathbf{\Phi}}_{bn})^t \qquad (2)\end{aligned}$$

where $\mathbf{C}_q^p[k+1,l+1] = <\overline{\phi}_p^k, \phi_p^l>$.

Proof: Given the first $m \times n$ elements of $\{\overline{\mathbf{\Xi}}_{ab}^{kl}\}$ that expand a subspace of the same dimension as the one expanded by the first $m \times n$ elements of $\{\mathbf{\Xi}_{ab}^{kl}\}$, it can be shown that

$$\overline{\mathbf{\Omega}}_{mn} = ((\mathbf{\Phi}_{am})^t\ \overline{\mathbf{\Phi}}_{am})^{-1}\ \mathbf{\Omega}_{mn}\ ((\overline{\mathbf{\Phi}}_{bn})^t\ \mathbf{\Phi}_{bn})^{-1},$$

where $\mathbf{\Omega}_{mn} = (\mathbf{\Phi}_{am})^t\ \mathbf{I}_{ab}\ \mathbf{\Phi}_{bn}$ and $\overline{\mathbf{\Omega}}_{mn} = (\overline{\mathbf{\Phi}}_{am})^t\ \mathbf{I}_{ab}\ \overline{\mathbf{\Phi}}_{bn}$. Then, the lemma follows straightforwardly. ■

Corollary 2: If the subsets of $\{\mathbf{\Xi}_{ab}^{kl}\}$ and $\{\overline{\mathbf{\Xi}}_{ab}^{kl}\}$ span the same subspace, then $\hat{\mathbf{I}}_{ab}^{mn} = \overline{\mathbf{I}}_{ab}^{mn}$.

Lemmas 1 and 2 are the key elements for the new reconstruction method but, before introducing it, let us reformulate the standard least-squares method in terms of the formalism just introduced.

III. Revisiting the standard method

The geometric moment of order (m,n) with respect to the origin of image $\mathbf{I}_{ab}$ is defined as:

$$\mu_{mn} = \sum_{x=1}^{a}\sum_{y=1}^{b} x^m y^n \mathbf{I}_{ab}[x,y].$$

Then, they can be seen as the projection coefficients of the image onto the basis set of monomial functions, that is $\mathbf{\Omega}_{mn}[k+1,l+1] = \mu_{kl}$, $\mathbf{\Xi}_{mn}^{kl}[x+1,y+1] = x^k y^l$, and $\mathbf{\Phi}_{pq}[k,l] = k^{l-1}$, for $k = 0,\ldots,m-1$ and $l = 0,\ldots,n-1$.

As a consequence, in our case, $\mathbf{\Omega}_{mn}$ are *Vandermonde matrices* and their associated Gram matrices correspond to what are known as *Hilbert matrices* whose general term is [11]:

$$\mathbf{\Gamma}_q^p[k+1,l+1] = \frac{1}{k+l+1}.$$

According to Corollary 2, if both the projection basis, $\{\mathbf{\Xi}_{ab}^{kl}\}$, and the orthogonal reconstruction basis, $\{\overline{\mathbf{\Xi}}_{ab}^{kl}\}$, span the same subspace, then the reconstructed images obtained, using either Lemma 1 or Lemma 2, coincide.

In figure 1, the reconstruction of the binary pattern "E" is carried out using the result of Lemma 2 and taking as basis $\{\overline{\mathbf{\Xi}}_{ab}^{kl}\}$ the one corresponding to Tchebichef

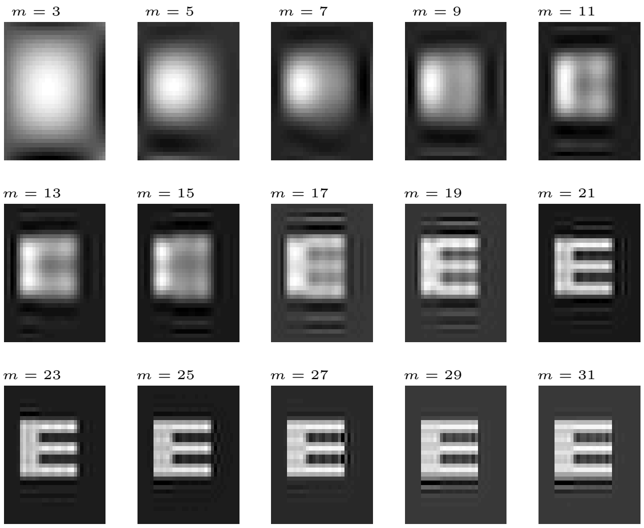

Fig. 1. Reconstructed 32×32 images, from its moments up to order (m, m), using Lemma 2 and Tchebichef polynomials as orthogonal polynomial basis.

polynomials. The same results are obtained if Lemma 1 is used instead, i.e. if pseudoinverses of Vandermonde matrices are computed.

Note that ill-posedness is solved here by imposing null value to those coefficients of the orthogonal basis set used in the reconstruction that have higher order than the maximum order of available moments. Next, it is shown how the use of other orthogonal basis sets different from polynomials allow to solve ill-posedness by introducing constraints directly on the image characteristics such as its bandwidth or spatial resolution.

IV. A Novel Reconstruction Method

The application of Lemma 2 leads to a reconstruction of the image by its truncated series expansion onto an orthonormal basis set. In this section we explore the possibility that the projection and the reconstruction subspaces are not the same. To this end, the coefficients associated with those reconstruction functions of order higher than the maximum order of available moments are assumed to be null. All the others are obtained from the available moments.

A. *Reconstructing a band-limited image*

In terms of the Fourier transform coefficients of the image, the band-limiting assumption means that Fourier coefficients of order greater or equal to (m, n) are null.

Fourier coefficients are normally defined as

$$c_{k,l} = \frac{1}{\sqrt{ab}} \sum_{x=1}^{a} \sum_{y=1}^{b} \mathbf{I}_{ab}\, \mathrm{e}^{-j2\pi\left(\frac{(x-1)(k-1)}{a} + \frac{(y-1)(l-1)}{b}\right)}.$$

Nevertheless, a relocation of these coefficients in matrix $\mathbf{C}_{mn}$ is carried out here so that increasing indexes correspond to higher frequency coefficients. In this case,

$$\mathbf{C}_{mn}[k,l] = \frac{1}{\sqrt{ab}} \sum_{x=1}^{a} \sum_{y=1}^{b} \mathbf{I}_{ab}\, \mathrm{e}^{-j2\pi\left(\frac{(x-1)(k-\frac{(m-1)}{2}-1)}{a} + \frac{(y-\frac{(n-1)}{2}-1)(l-1)}{b}\right)}.$$

Then, these Fourier coefficients can be seen as the projection coefficients of the image onto complex exponential functions of the form $\overline{\mathbf{\Phi}}_{pq}[k,l] = \frac{1}{p}\,\mathrm{e}^{-j2\pi\left(\frac{(k-1)(l-\frac{(q-1)}{2}-1)}{p}\right)}$. Substituting these orthogonal basis matrices in the result of Lemma 2, a low-pass approximation of the original image is obtained from a set of its geometric moments. Figure 2 shows the obtained results using the same moments as in figure 1.

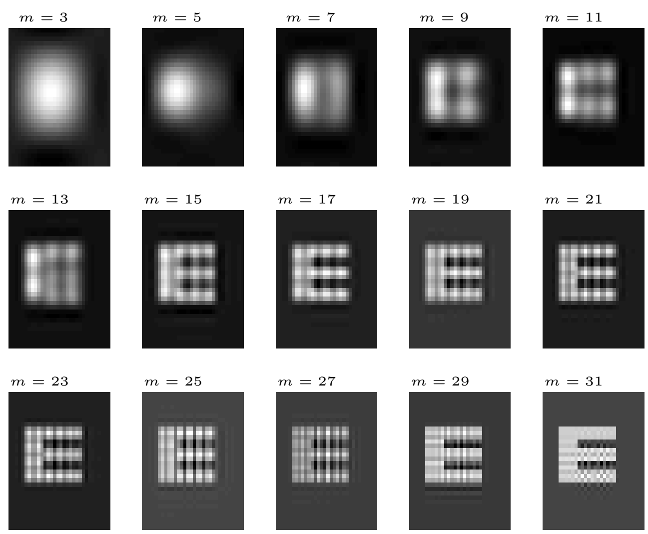

Fig. 2. *Band-limited reconstruction of a 32×32 image, from its moments up to order (m, m), using Lemma 2 and Fourier coefficients.*

Note that a band-limited image can be perfectly reconstructed, using this method, when the number of moments obtained from the image is equal or greater than the number of its significant discrete Fourier spectrum coefficients.

Likewise, a high-pass approximation could be obtained if the basis function associated with higher frequencies were considered instead.

B. *Reconstructing a resolution-limited image*

Limiting the resolution of an image means eliminating those regions of smaller size than a given one. In terms of the Haar transform, this requirement becomes trivial since its main characteristic is the direct relationship between the number of coefficients and the spatial resolution of the image.

Haar coefficients are obtained from the projection of the image onto the Haar functions $h_k^N(z)$, which are defined over the closed interval $z \in [0,1]$ and for $k = 0, 1, 2, \ldots, n-1$, where $N = 2^n$, as:

$$h_0^N(z) = h_{00}^N(z) = \frac{1}{\sqrt{N}},$$

and

$$h_k^N(z) = h_{pq}^N(z) = \frac{1}{\sqrt{N}} \begin{cases} 2^{\frac{p}{2}} & \text{if } \frac{q-1}{2^p} \le z < \frac{q-\frac{1}{2}}{2^p}, \\ -2^{\frac{p}{2}} & \text{if } \frac{q-\frac{1}{2}}{2^p} \le z < \frac{q}{2^p}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\overline{\mathbf{\Phi}}_{pq}[k,l] = h_l^p(k)$. Substituting these orthogonal basis matrices in the result of Lemma 2, a resolution-limited approximation of the original image is obtained from a set of its geometric moments. Figure 3 shows the obtained results for the same pattern used in figure 1.

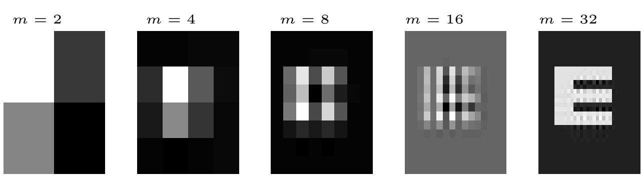

Fig. 3. Resolution-limited reconstruction of a 32×32 image, from its moments up to order (m, m), using Lemma 2 and Haar coefficients.

Note that a resolution-limited image can be perfectly reconstructed when the number of moments obtained from it is equal or higher than the number of its significant discrete Haar coefficients.

V. Numerical conditioning

The application of Lemma 1 requires the computation of pseudoinverses and hence the inversion of Gram matrices which can be an ill-conditioned problem for some projection basis. In particular, reconstructing an image from a set of its geometric moments is an ill-conditioned problem, i.e., small perturbations in the data generate large errors in the reconstructed image which prevent us from obtaining an effective solution. This is what is usually assumed by the image processing community. Nevertheless, totally positive Vandermonde systems, as it is our case, can be solved very accurately, regardless of their condition number, using Björck-Pereyra-type methods [1]. Actually, the ordinary definition of condition number is not adequate for describing the numerical conditioning of positive linear systems whose initial minors can be computed accurately and the true condition number is much smaller [3], as it can be easily shown in our case. Björck-Pereyra methods are the perfect example of structure exploiting algorithms that deliver more accuracy than traditional algorithms and run in time $O(n^2)$ compared with $O(n^3)$ for the traditional algorithms.

VI. Conclusions

A desirable property for the basis functions of the series approximation of an image is that they concentrate most of the information in a reduced amount of coefficients. What information means depends on the interpretation of the basis; however, most common applications refer to bandwidth or spatial resolution, which are associated with Fourier and Haar coefficients, respectively. Then, setting a relationship between these coefficients and moments provides a straightforward interpretation of the information contained in moments, as well as a method for reconstructing an image from a given set of moments. None of the former methods provided the proper setting to introduce these constraints.

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