Propagation and Fusion of Uncertain Geometric Information *

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Abstract

Propagation and fusion of geometric information is of great significance in multisensorial systems, mainly in robotics applications, where multiple sensors or mobile sensor systems that change their perspective of the environment capture uncertain sparse, and sometimes partial, geometric data. In a sensor data fusion problem a set of constraints that describe the relationships between problem inputs and desired solutions can be defined. Constraints and geometric features can be organized in a graph in which nodes stand for geometric primitives – whose uncertainty in their location is represented by regions in their parameter spaces – and arcs for constraints. This paper deals with the problem of propagating uncertainty sets over graphs of geometric constraints. When a new measurement is acquired, a new uncertainty set is introduced for the corresponding geometric feature. This set is propagated all over the graph of geometric constraints and fused at each node with previous information, updated sets are thus obtained as well as final uncertainty regions for each feature.

1. Introduction

The increasing requirements of adaptability and autonomy in robotics demand the ability of a system to react to sensor data. This is particularly true when the system must operate in unstructured environments. However, unstructured situations pose challenging problems in sensor data fusion and sensor-based decision making under uncertainty.

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Basically, two approaches have been used, in the robotics domain, to model uncertainties: (a) tolerance limits, worst case or *set membership approach* [Br], [AH], [DP89],[ST]; and (b) multidimensional probability distributions or *stochastic approach* [Du], [N],[P], [OHF], [SSC]. The latter assumes gaussian distribution for errors thus representing uncertainties with a covariance matrix. The former describes them with an uncertainty vector that puts bounds on the parameters thus defining an uncertainty set in the space of parameters where the actual value is bound to be. The set membership approach is adopted here because it makes no assumption about the nature of sensing errors, being well-suited for robotics applications where most important errors come from quantization. Moreover, contrary to the stochastic approach, it avoids the general assumptions of unbiased and independent measurements.

The underlying set membership principles have been applied in the control and systems science domains. In this context, two main sorts of uncertainty sets have been used: polytopes [MB], [WP] and ellipsoids [De], [F].

In unstructured environments, particularly when geometric issues are involved, a set of constraints derived from stored models or from the incoming data itself that are physically plausible and solvable will be useful for sensor data integration. We assume that we are able to obtain this set of constraints and we will concentrate ourselves on the propagation and fusion of uncertain information using those constraints.

This paper is structured as follows. Section 2 tackles the problem of finding a good representation for geometric features and their symmetries. Section 3 describes the adopted uncertainty model. Section 4 deals with the problem of propagating information through geometric constraints. Finally, an example is provided in Section 5.

2. Representation of Geometric Features and Their Symmetries

In this section, we describe a representation, already introduced in [SSC] and [T], based on local reference frames that simplifies later treatments.

Any geometric feature has a local reference frame attached to it. The location and orientation of a geometric feature is represented by the transformation (rigid motion) from the global reference frame to its local frame. A transformation t_{AB} from frame A to frame B is represented using a vector $\mathbf{x}_{AB}$ composed by the three cartesian coordinates for location and the Roll-Pitch-Yaw angles for orientation, which is called *location vector*. In other words,

$$t_{AB} = Trans(x,y,z) \cdot Rot_z(\phi) \cdot Rot_y(\theta) \cdot Rot_x(\psi), \quad (1)$$

$\mathbf{x}_{AB} = (x,y,z,\phi,\theta,\psi)^t$ being the location vector. Representing geometric features by local references has the advantage of homogeneity: all types of elements are described by the same kind of parameters. Moreover, data can be expressed in any reference (world frame, local frame, sensor frame, object frame, etc.) and reference changes can be performed whenever they are needed.

Many geometric features have intrinsic symmetries. For instance, a line in space has translational and rotational symmetries along and around itself. Any value is valid for a symmetry parameter in the location vector, because them all define possible local references for the feature. Geometric symmetries are represented by the corresponding *symmetry* of an element.

Definition 1. *Symmetry set.* Given a geometric feature defined by a set of points $E \subseteq \Re^3$, its symmetry set is the set of transformations that preserve E.

Symmetry sets are subgroups of the $\Re^3 \times SO(3)$, the group of positive rigid motions in $\Re^3$ with composition operation. Symmetry groups of elements of the same type are conjugated and thus isomorphic. Symmetry subgroups of elements of the same type are equal if they are defined in their respective local frames.

Local references for each type of feature will be chosen according to its geometric symmetry, so that symmetry groups have simple expressions. A symmetry group S can be expressed as $S = M \cdot C$, where M is a group of continuous motions and C is a finite group of cyclic motions [Bu]. A subgroup M of continuous motions can be expressed using the corresponding location vectors and a specific kind of matrices, the *binding matrices* [T]. A *binding matrix* B is a matrix formed by n rows of the identity matrix I_6, taken in order, with $n \leq 6$. According to this definition

$$M = \{t \in \Re^3 \times SO(3) \mid B \cdot \mathbf{x}(t) = 0\}. \quad (2)$$

Some of the coordinates of $\mathbf{x}(t)$ do not appear in the equation $B \cdot \mathbf{x}(t) = 0$, they correspond to the degrees of freedom of the feature, which will be called *free coordinates*, and the others, *assigned* coordinates. For example, the continuous motion group M of a line in space is

$$M = T_x \cdot R_x = \{t \mid \mathbf{x}(t) = (x,y,z,\phi,\theta,\psi) \text{ and } y = z = \phi = \theta = 0\} =$$
$$= \left\{ t \mid \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} (3)$$

See [T] for a complete list of local references, symmetry groups and binding matrices for different types of geometric features.

3. Uncertainty Model

3.1. Perturbation Vectors

A sensory observation of a geometric feature provides an approximate mesure t_{app} of the actual transformation t_{act} between the world reference and the feature reference. Therefore,

$$t_{act} = t_{app} \circ t_{error}, \quad (4)$$

or equivalently, $t_{WE} = t_{WE'} \circ t_{E'E}$, where the transformation t_{error} (or $t_{E'E}$) is supposed to be *small*, that is, *near* the identity transformation. Notice that t_{error} is expressed in local reference E', not in the world reference. Thus, when world frame is changed, t_{app} has to be updated, but t_{error} remains the same.

Equation (4) leads to

$$\mathbf{x}_{act} = \mathbf{x}_{app} \oplus \mathbf{x}_{error} \quad (5)$$

where $\oplus$ is the composition operator for location vectors [SSC].

Since free coordinates are given in local coordinate frames of the location vector $\mathbf{x}_{error}$, they can be considered exactly known and, since any value is valid for them, we will assume they are zero. Hence, only assigned coordinates have associated uncertainty [P].

Definition 2. *Perturbation vector* [T]. Given an incremental vector $\mathbf{x}_{error}$ of a geometric feature, the corresponding *perturbation vector* $\mathbf{p}$ is made up the assigned coordinates of $\mathbf{x}_{error}$.

It is easy to prove that $\mathbf{p} = B \cdot \mathbf{x}_{error}$, where B is the binding matrix of the element. In addition, $\mathbf{x}_{error} = B^t \mathbf{p}$ and then equation (5) can be rewritten as

$$\mathbf{x}_{act} = \mathbf{x}_{app} \oplus B^t \mathbf{p}. \tag{6}$$

3.2. Set Membership Approach to Uncertainty Manipulation

A set membership approach is used here to model uncertainty for perturbation vectors, so that the only information known about an uncertain vector $\mathbf{p}$ is a bounding set $\mathscr{U}_{\mathbf{p}}$ called *uncertainty region*. The actual value of $\mathbf{p}$ is known to lie inside $\mathscr{U}_{\mathbf{p}}$ and any point in $\mathscr{U}_{\mathbf{p}}$ is valid as an estimation of $\mathbf{p}$ [ST]. An observation of a geometric feature will consist of a pair $(\mathbf{x}, \mathscr{U})$ where $\mathbf{x}$ is the nominal location vector with respect to the world reference frame and $\mathscr{U}$ is the uncertainty region for the perturbation vector.

Sensor errors can be approximately described as bounds in each of the coordinates of the parameter vector describing a feature, so that they are linear and define halfspaces or strips in the parameter space. Uncertainty regions are the intersection set of all these halfspaces leading to polytopes. Uncertainty polytopes can also be approximated by ellipsoidal sets (ellipsoids or ellipsoidal cylinders) containing them. Ellipsoidal sets are represented by a vector (center) and a positive or semipositive defined matrix, reducing computations to matrix operations and decreasing the amount of storage. While simple, ellipsoidal sets still model the different uncertainty of the observations in different directions of the parameter space.

Coordinates of perturbations vectors are not overdetermined, they are never exactly known. Thus, uncertainty sets have always maximum dimension, that is they are not contained in any hyperplane. This means that matrices representing ellipsoidal sets are always well defined.

Information of a geometric feature given by an observation can be complete or partial. Uncertainty regions of complete observations are bounded regions in $\Re^n$, where n is the number of degrees of freedom of the sensed feature. That is, they will be bounded polytopes or non-degenerate ellipsoids of dimension n. On the contrary, uncertainty sets of partial observations are sets of $\Re^n$ unbounded along one or more directions. Unbounded directions are those in which we do not have any information. The number of these directions is $n - r$, where r is the number of independent relations between the coordinates. Therefore, uncertainty regions for partial information are unbounded polytopes or ellipsoidal cylinders.

3.3. Changing References

Proposition 1. Given an observation of an element with respect to a reference W by $(\mathbf{x}_{WE}, \mathscr{U}_E)$ and an observation of the W reference location with respect to a reference V by $(\mathbf{x}_{VW}, \mathscr{U}_W)$, then the location of the element with respect to V is given by $(\mathbf{x}_{VE}, \hat{\mathscr{U}}_E)$ where $\mathbf{x}_{VE} = \mathbf{x}_{VW} \oplus \mathbf{x}_{WE}$, and $\hat{\mathscr{U}}_E$ is the image set of $\mathscr{U}_W \times \mathscr{U}_E$ by the linear map

$$(\mathbf{p}_W, \mathbf{p}_E) \mapsto J\mathbf{p}_W + \mathbf{p}_E \tag{7}$$

where

$$J = B_E (J_{2\oplus}(\mathbf{x}_{WE}, 0))^{-1} J_{1\oplus}(0, \mathbf{x}_{WE}) B_W^t,$$

$J_{1\oplus}(x,y)$ and $J_{2\oplus}(x,y)$ being the partial derivatives of $x \oplus y$.

Proof. Equation (6) for actual $\mathbf{x}_{VW}$ and ${}_{WE}$ leads to

$$\mathbf{x}_{VE}^{act} = \mathbf{x}_{VW} \oplus \mathbf{x}_{WE} \oplus B_E^t B_E (\ominus \mathbf{x}_{WE} \oplus B_W^t \mathbf{p}_W \oplus \mathbf{x}_{WE} \oplus B_E^t \mathbf{p}_E) \tag{8}$$

where $\ominus$ is the location vectors operator corresponding to the inverse transformation. The above equation mean that the new nominal vector is

$$\mathbf{x}_{VE} = \mathbf{x}_{VW} \oplus \mathbf{x}_{WE}, \tag{9}$$

and the new perturbation vector is

$$\mathbf{p}_E' = B_E (\ominus \mathbf{x}_{WE} \oplus B_W^t \mathbf{p}_W \oplus \mathbf{x}_{WE} \oplus B_E^t \mathbf{p}_E). \tag{10}$$

Linearizing around $\mathbf{p}_E = 0$, $\mathbf{p}_W = 0$ we get $\mathbf{p}_E' = J\mathbf{p}_W + \mathbf{p}_E$. $\square$

When using polytopes as uncertainty sets, $\mathscr{U}_W \times \mathscr{U}_E$ will be a polytope and so $\hat{\mathscr{U}}_E$ can be exactly obtained as a projection computation [DP90]. If $\mathscr{U}_W$ and $\mathscr{U}_E$ are ellipsoids $\mathscr{U}_W \times \mathscr{U}_E$ is not an ellipsoid, but the smallest ellipsoid $\mathscr{U}_{WE}$ containing it can be computed by simple matrix operations. Then the linear image set of the ellipsoid $\mathscr{U}_{WE}$ will be an ellipsoid (see section 4.1) and can be taken as an ellipsoid approximating $\hat{\mathscr{U}}_E$. In addition, if $\mathscr{U}_E$ and $\mathscr{U}_W$ are the best ellipsoidal fit for polytopes $\mathscr{P}_E$ and $\mathscr{P}_W$, the resulting ellipsoid $\mathscr{U}_{WE}$ is the best ellipsoidal fit for $\mathscr{P}_E \times \mathscr{P}_W$.

There are two noticeable particular cases of proposition 1. Firstly, when changing world reference, exact location of reference W with respect to V is known. In this case, since there is no uncertainty on W, $J = 0$ and $\hat{\mathscr{U}}_E = \mathscr{U}_E$. Secondly, when exact location of a feature with respect to W is available, $\hat{\mathscr{U}}_E$ is the image set of $\mathscr{U}_W$ by the linear map $\mathbf{p}_W \mapsto J\mathbf{p}_W$, avoiding the computation of $\mathscr{U}_W \times \mathscr{U}_E$.

3.4. Fusion of Competitive Information

Since the sensory information of geometric features are uncertain, multiple competitive observations of the same feature have to be combined to obtain a good estimation. Different measurements, each one with its own uncertainty are fused to obtain a unique estimation. Uncertainty of the resulting estimation is also required.

In the set membership approach actual data is known to lie inside all the uncertainty sets of the redundant measurements to be fused, consequently intersection of the uncertainty sets is needed. When dealing with polytopic sets, Computational Geometry techniques are used to compute intersection of polytopes [BS, PW]. If the ellipsoidal approach is used, the fusion ellipsoid is computed in several steps, each one computing the smallest ellipsoid containing the intersection of the previous estimation ellipsoid and a halfspace (or a strip). This is directly carried out through simple matrix computations [ST, WP]. This fusion technique allows fusion of complete information as well as partial information.

Tests for detecting inconsistent and redundant measurements can be easily implemented. If two measurements are inconsistent, their uncertainty sets are disjoint. If two measurements are redundant, one uncertainty region associated with one measurement is totally contained in the other one.

As it has been already mentioned, one of the main advantages of the set membership approach is that independency between observations to be fused is not required. This is important because the propagation algorithm described in next section is based on this fact.

4. Geometric Constraints

A point lying on a line, two parallel lines, a plane containing a point, etc. are spatial relationships which can be expressed as geometric constraints. Since information about a feature gives information about other features related to it through constraints, constraints allow to transfer information. Also, relations between sensed elements can be checked for consistency using this idea.

A geometric constraint between two geometric constraints $\mathscr{E}$ and $\mathscr{F}$ is given by functional vectorial equation $h(\mathbf{p},\mathbf{q}) = 0$ between their perturbation vectors $\mathbf{p}$ and $\mathbf{q}$. Relations between the uncertainty sets can be computed through the vectorial equations.

Let us suppose that features $\mathscr{E}$ and $\mathscr{F}$ satisfy a given constraint, say point $\mathscr{F}$ lies on line $\mathscr{E}$. Then, the transformation t_{EF} between them has to satisfy some parameter constraints. Due to the way local references have been chosen, most of the constraints can be expressed using binding matrices, that is, $\mathscr{E}$ is related with $\mathscr{F}$ if and only if

$$B \cdot \mathbf{x}_{EF} = 0 \tag{11}$$

where B is a binding matrix.

In our example point $\mathscr{F}$ lies on line $\mathscr{E}$ if and only if $t_{EF} \in T_x R_{xyz}$ or, in other words,

$$\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad 0 \right) \mathbf{x}_{EF} = 0. \tag{12}$$

Equation (11) can be written in terms of perturbation vectors of $\mathscr{E}$ and $\mathscr{F}$, leading to

$$B(\ominus B_E^t \mathbf{p} \oplus \hat{\mathbf{x}}_{EF} \oplus B_F^t \mathbf{q}) = 0, \tag{13}$$

where $\hat{\mathbf{x}}_{EF}$ is an estimation of $\mathbf{x}_{EF}$ and B_E and B_F are the binding matrices of features $\mathscr{E}$ and $\mathscr{F}$.

Only linear equations allow effective computation for the related uncertainty sets, therefore non linear equations must be linearized. Since $\mathbf{p}$ and $\mathbf{q}$ are assumed to be small, equation (13) can be linearized around $\mathbf{p} = 0$ and $\mathbf{q} = 0$.

4.1. Transferring Uncertainties in Local References

The information about the perturbation vector $\mathbf{q}$ obtained from the information about $\mathbf{p}$, using a constraint between them, will be either complete or partial, depending on the number of independent relations (range of h) which will be always less or equal than dimension of vector $\mathbf{q}$.

Given a linear constraint $h(\mathbf{p},\mathbf{q}) = 0$ between two elements and the uncertainty set of one of them, say $\mathscr{U}_{\mathbf{p}}$, the corresponding uncertainty set $\mathscr{U}_{\mathbf{q}}$ for the other can be computed. Transformation of polytopes through linear relations can be carried out using Computational Geometry techniques [DP90]. Transformation of ellipsoids is computed using the following proposition.

Proposition 2. Given an ellipsoid $\mathscr{U}_{\mathbf{p}}$with center $\mathbf{p}_0$ and matrix E_p, the image ellipsoid through the linear relation $A\mathbf{p}+C\mathbf{q}+D=0$ has center $\mathbf{q}_0$ such that $A\mathbf{p}_0+C\mathbf{q}_0+D=0$ and matrix

$$E_{\mathbf{q}}=C^t\left(AE_{\mathbf{p}}^{-1}A^t\right)^{-1}C. \tag{14}$$

Proof. Let us define $\mathbf{r}=-A\mathbf{p}=C\mathbf{q}+D$. The projection of ellipsoid $\mathscr{U}_{\mathbf{p}}$ is obtained through matrix operations, getting an ellipsoid $\mathscr{U}_{\mathbf{r}}$ with matrix $E_{\mathbf{r}}=(AE_{\mathbf{p}}^{-1}A^t)^{-1}$ and center $\mathbf{r}_0=-A\mathbf{p}_0$. Simple substitution of $\mathbf{r}=C\mathbf{q}+D$ into $\mathscr{U}_{\mathbf{r}}$ equation, gives the equation for $\mathscr{U}_{\mathbf{q}}$. $\square$

4.2. Checking Relations between Sensed Features

Given two geometric features with perturbation vectors $\mathbf{p}$ and $\mathbf{q}$ and associated uncertainty sets $\mathscr{U}_{\mathbf{p}}$ and $\mathscr{U}_{\mathbf{q}}$, respectively, we are interested in checking whether they satisfy the equation $h(\mathbf{p},\mathbf{q})=0$. An uncertainty set $\mathscr{U}_{(\mathbf{p},\mathbf{q})}$ for $(\mathbf{p},\mathbf{q})$ can be easily computed, from sets $\mathscr{U}_{\mathbf{p}}$ and $\mathscr{U}_{\mathbf{q}}$, using matrix operations. Then, using the proposition in previous section, an uncertainty set $\mathscr{U}_h$ can be derived for $h=h(\mathbf{p},\mathbf{q})$. The equation $h(\mathbf{p},\mathbf{q})=0$ is said to be *uncertainly satisfied* with uncertainty $\mathscr{U}_h$ if $0\in\mathscr{U}_h$.

Uncertain satisfaction arises when performing recognition tasks with uncertain data. Recognition methods require to check a lot of relations, such as two features are the same, a point lies on a line, angles between faces are the same, etc.

In practice, since most relation checking have negative response, a previous test to reject clearly unsatisfactory cases for ellipsoidal uncertainty sets can be carry out [OHF].

A point $\mathbf{p}$ is inside ellipsoid $\mathscr{U}$ with center $\mathbf{p}_0$ and inverse matrix M, if

$$(\mathbf{p}-\mathbf{p}_0)^t M^{-1}(\mathbf{p}-\mathbf{p}_0)\leq 1. \tag{15}$$

On the other hand, it can be shown that

$$\frac{1}{\text{trace }M}(\mathbf{p}-\mathbf{p}_0)^t(\mathbf{p}-\mathbf{p}_0)\leq(\mathbf{p}-\mathbf{p}_0)^t M^{-1}(\mathbf{p}-\mathbf{p}_0) \tag{16}$$

Thus, if

$$\frac{1}{\text{trace }M}(\mathbf{p}-\mathbf{p}_0)^t(\mathbf{p}-\mathbf{p}_0)\geq 1, \tag{17}$$

inequality (15) will not be satisfied, avoiding further computations thus reducing computational cost.

4.3. Graph of Geometric Constraints

A graph of geometric constraints has been defined as a graph whose nodes stand for geometric features and edges stand for geometric constraints. Unlike the relation graphs defined in [Du] and [SSC], arcs of the graph adopted here are associated with geometric constraints (known without uncertainty), that is, they do not stand for uncertain spatial relations between features.

Each time a new sensory observation of an element is acquired, the information should be propagated to all other nodes in the graph through the edges. In each node when a new information is arrived (coming directly from a sensor or from an other node), it will be transferred to its neighbour nodes and later fused with the previous information to obtain a new uncertainty set.

Information from one node to all the others should be propagated through all the possible paths in the graph, because some edges may give different information than the others. However, paths followed should not form a loop because an observation of a node coming from itself gives no new information. Propagation paths are cut down when some of the following situations arise: *i*) a terminal node is reached, *ii*) new information does not improve previous information about the node and *iii*) uncertainty volume of new information is too large (there is no profit on propagating very poor information). Propagation is carried out by the following recursive algorithm where each edge has been replaced by two directed edges (with opposite directions) and all edges are assumed to be initially marked as 'non visited'.

algorithm Propagation
    input node, $\mathscr{U}_{\text{node}}^{\text{new}}$
    create an imaginary edge leaving and arriving at the node
        and having the identity functional relations
    PropagateThrough (imaginary edge)
endalg

subalgorithm PropagateThrough (edge)
    nodei= initial node of the edge
    nodef= final node of the edge
    $\mathscr{U}_{\text{nodef}}^{\text{new}}$=Transformation of $\mathscr{U}_{\text{nodei}}^{\text{new}}$ using the edge relations
    if InconsistencyTest ($\mathscr{U}_{\text{nodef}}^{\text{new}}$, $\mathscr{U}_{\text{nodef}}$) = YES *then* *exit*
    if ImproveTest ($\mathscr{U}_{\text{nodef}}^{\text{new}}$, $\mathscr{U}_{\text{nodef}}$) = NO *then* *exit*
    if VolumeTest ($\mathscr{U}_{\text{nodef}}^{\text{new}}$)= 'too large' *then* *exit*
    $\mathscr{U}_{\text{nodef}}=$ Fusion($\mathscr{U}_{\text{nodef}}$, $\mathscr{U}_{\text{nodef}}^{new}$)
    for each edge arriving to nodef

mark it as ‘visited’

endfor

for each edge t leaving from nodef

if visited(t) = NO *then* PropagateThrough (t)

endif

endfor

for each edge arriving at nodef

mark it as ‘non visited’

endfor

endsubalg

In the above procedure the propagation of an observation from node *A* many new observations for any other node *B*, one for each path from *A* to *B*. All the new observations in *B* will be fused with its previous estimation. Therefore the fusion technique used need to allow non-independent measurements, since in the case described they are clearly dependent.

The presented propagation procedure ensures the consistency of the graph, in the sense that uncertainty sets in all nodes are consistent.

5. Example

In the object of fig. 1 vertices v_1, v_2, v_3 and v_4, edges e_1 and e_2 and faces f_1 and f_2 are considered. Incidence constraints stored in the model define the corresponding graph of geometric constraints (fig. 2).

Graph arcs of type (v_i, f_j) stand for relation vertex v_i *in* face f_j and have vectorial equation of form (13). The estimation $\hat{\mathbf{x}}_{f_j v_i}$ can be taken satisfying $B_{fv}\hat{\mathbf{x}}_{f_j v_i} = 0$, since actual $\mathbf{x}_{f_j v_i}$ satisfies this constraint. Therefore, linearizing equation (13), we get

$$B_{fv}\left(J_{2\oplus}(\hat{\mathbf{x}}_{f_j v_i},0)B_v^t\mathbf{p}_{v_i} - J_{1\oplus}(0,\hat{\mathbf{x}}_{f_j v_i})B_f^t\mathbf{p}_{f_j}\right) = 0, \tag{18}$$

where

$$B_{fv} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_v = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad 0 \right),$$

$$B_f = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{pmatrix},$$

$$\mathbf{p}_v = (x_v, y_v, z_v)^t, \quad \text{and} \quad \mathbf{p}_f = (z_f, \phi_f, \theta_f)^t$$

Likewise, constraints of type (v_i, e_j) can be expressed as

$$B_{ev}\left(J_{2\oplus}(\hat{\mathbf{x}}_{e_j v_i},0)B_v^t\mathbf{p}_{v_i} - J_{1\oplus}(0,\hat{\mathbf{x}}_{e_j v_i})B_e^t\mathbf{p}_{e_j}\right) = 0 \tag{19}$$

where

$$B_{ev} = \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad 0 \right), \quad B_e = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{pmatrix}$$

and

$$\mathbf{p}_e = (y_e, z_e, \phi_e, \theta_e)^t$$

Finally, constraints of type (e_i, f_j) can be expressed as

$$B_{fe}\left(J_{2\oplus}(\hat{\mathbf{x}}_{f_j e_i},0)B_e^t\mathbf{p}_{e_i} - J_{1\oplus}(0,\hat{\mathbf{x}}_{f_j e_i})B_f^t\mathbf{p}_{f_j}\right) = 0 \tag{20}$$

where

$$B_{fe} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

In the proposed experiment uncertain sensory data are available for elements v_1, v_2, v_3, v_4, f_1 and f_2, but not for e_1 and e_2. With the above explained graph propagation procedure more accurate estimations of the former features are obtained, as well as estimations for edges e_1 and e_2. In the final paper numerical results will be provided showing uncertainty volumes before and after propagation.

Estimations and uncertainties for edges e_1 and e_2 are derived from estimations of other features, with no direct sensory acquisition. Situations like this one show the great profit that can be obtained from propagation technique. Also the resulting estimations may state, with small uncertainty, that both edges are collinear, using the methods presented in section 4.2. The collinearity constraint in vectorial linear form leads to

$$h(\mathbf{p}_1, \mathbf{p}_2) =$$

$$= B_{ee}\left(\hat{\mathbf{x}}_{E_1E_2} + J_{2\oplus}(\hat{\mathbf{x}}_{E_1E_2},0)B_e^t\mathbf{p}_2 - J_{1\oplus}(0,\hat{\mathbf{x}}_{E_1E_2})B_e^t\mathbf{p}_1\right) = 0 \tag{21}$$

where

$$B_{ee} = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

Since 0 belongs to the uncertainty set of $h(\mathbf{p}_1, \mathbf{p}_2)$ and the volume of this set is under a threshold, it means that 0 is a good estimation for it, and so $e_1 = e_2$ is a valid statement.

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