New Algebraic Conditions for the Identification of the Relative Position of Two Coplanar Ellipses

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Abstract

The identification of the relative position of two real coplanar ellipses can be reduced to the identification of the nature of the singular conics in the pencil they define and, in general, their location with respect to these singular conics in the pencil. This latter problem reduces to find the relative location of the roots of univariate polynomials. Since it is usually desired that all generated expression are algebraic to simplify further analysis, including the case in which the ellipses undergone temporal variations, all recent methods available in the literature rely mathematical tools such as Sturm-Habicht sequences or subresultant sequences. This paper presents an alternative based on more elementary tools which results in a binary decision tree to classify the relative location of two ellipses in 12 different classes. The decision at each node is taken based on the sign of a set of algebraic/rational expressions on the ellipses coefficients, the most complex of them being third and second order polynomial discriminants.

Keywords: Ellipses, pencils of conics, interference detection, positional relationships.

1. Introduction

The problem of identifying the relative position of two ellipses arises in widely disparate fields that include, for example, robotics, computer vision, CAD/CAM, and computer animation. For example, ellipses are used in these fields to model (or enclose) the shape of planar objects [2] or planar uncertainty regions [3], or even to characterize the singularities of some robots [4].

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The positional relationship considered in this paper is the following: two pairs of realellipses have the *same relative position* if, and only if, they have the same type of intersection set in the complex projective plane, and the same nested-inside or disjoint-inside relationship, if any of the two applies. This gives the different positional relationships appearing in Fig. 1. Thus, the first step towards the classification of the relative position of two non-coincident real ellipses in the real affine plane is based on the analysis of their intersections in the complex projective plane, either real or imaginary (complex non-real). Relying on Levy's projective classification [5, Ch. IV, Sec. 11] of pencils of planar conics, we have the following nine possible intersection sets:

I : four simple intersection points:

- a : all imaginary;
- b : two real and two imaginary;
- c : all real;

II : one double intersection point and two simple intersection points:

- a : the two simple points are imaginary;
- b : the two simple points are real;

III : two double intersection points:

- a : all imaginary;
- b : all real;

IV : one triple and one simple intersection point, which are all real;

V : one quadruple intersection point, which is real.

Hence, the positional classification addressed here is a refinement of the real projective classification of pencils of conics, since the nested or disjoint inside relationship is also taken into account (this is clear from Table 1). The classification above, together with a procedure to detect when one ellipse is completely inside the other, is enough to unequivocally identify the positional relationships appearing in Fig.1, if we exclude the case in which both ellipses are coincident (type 0). Thus, the problem can essentially be reduced to compute the roots of the system formed by the two ellipse equations, which can be reduced in turn to compute the roots of a quartic [6]. Although Ferrari's formulas permit computing these roots, as it is done in [7], they are so complicated so as to motivate the search for alternative approaches that avoid the explicit computation of the intersection points. It is important to observe that, even if the intersection points are needed, a preprocessing step to identify the positional relationship is important to simplify their computation [7].

In most applications, it is not necessary to identify the twelve types in Fig. 1 but a subset of them. Some of these types can readily identified by using some classical tests. Well-known tests exist to decide, for example, if two ellipses are in contact, have a double tangency, or have an osculating contact. They can already be found in the eighteenth chapter of Salmon's *Treatise on Conic Sections*, whose first edition dates back to 1848 [8], or in a more accessible language in the fifteenth chapter of Casey's treatise on analytic geometry [9]. They are algebraic conditions on the ellipses' coefficients. It is clear that some of the types in Fig. 1 can readily be identified using these tests, but others cannot be discerned.

Two ellipses with solution set III_a (with repeated imaginary points) are projectively equal to concentric circles. Identifying this situation has important applications in computer vision [10, 11]. The standard approach in the identification of this situation consists in finding if both ellipses have a common self-polar triangle. The use of self-polar triangles to identify the relative position of two ellipses is a standard technique [12, pp.96-100], but unfortunately it reduces to compute the roots of cubics, something that we want to avoid.

A recurring idea to solve the problem in all its generality, without having to explicitly compute the intersection points, consists in transforming the two ellipses, using an affine map, to a circle centered at the origin and an axis-aligned ellipse. Since the spatial relationship between two ellipses remains invariant to affine transformations, the problem is thus greatly simplified. It actually can be reduced, as explained in [13], to compute the extreme points on the axis-aligned ellipse that are closest and farthest from the origin, and the outward normals at these points.

A more insightful approach consists in analyzing the pencil of conics generated by the two ellipses. A survey on the obtained results using this approach can be found in [14]. The origin of this approach can be traced back to the development of the aforementioned classic tests [8]. The system of conics which passes through a fixed set of four points (no three collinear) is called a *pencil of conics* (remember that an ellipse is completely determined by five points). The four common points — *i.e.*, the intersection points between both ellipses, either real or imaginary— are called the *base points* of the pencil. An important result is that the pencil generated by two conics contains three degenerate conics consisting of real or imaginary pairs of lines and, what is even more important in our case, that it is possible to decide the nature of these degenerate conics for the nine types of solution sets enumerated above [5, p. 257]. The result is summarized in Table 1. In the first column we have the eleven possible positional relationships; the second contains the number and multiplicity of the real base points; the third, the kind of intersection set for both ellipses; and the fourth, the kind of degenerate conics in the corresponding pencil. Observe that the set of degenerate conics unequivocally identifies the kind of intersection set.

It is possible to combine the above two approaches. That is, it is possible to apply first an affine transformation to reduce the problem to that of computing the spatial relationship between a circle and an axis-aligned ellipse, and then analyze the pencil of conics they define. This is the approach followed in [17, 18].

A further refinement in the approach based on the analysis of the pencil consists in using Sturm-Habicht sequences, as explained in [19], where ten positional relationships which do not exactly match those in Fig. 1 are considered (for example, the osculating and hyperosculating contact are considered as a single case). One important feature of this approach is that it permits the manipulation of the derived formulae for the cases where the considered ellipses depend on one parameter. More recently, a complete systematization of the problem is finally provided in [15], where it is also generalized to find the relative position of two arbitrary real conics, not necessary ellipses. The approach is based on the characterization of the orbits of pencils of conics using classical invariant theory, and the characterization of the rigid isotropy class for each orbit. This requires the use of Descartes' law of signs and subresultant sequences. Building upon these results, the invariant theory of pencils of conics is further explored in [16] to show that some of the conditions obtained in [15] have unnecessary high degrees.

In this paper, we continue exploring the properties of the pencil of conics defined by two ellipses to come up with new algebraic conditions to decide the nature of the three degenerate conics it contains. In particular, we show how this problem reduces to study the nature of the roots of a cubic polynomial and how they are distributed with respect to the roots of two quadratics. The important point is to perform these operations without explicitly computing the roots of the cubic, otherwise the complexity of the approach would be equivalent to that of computing the intersection points of both ellipses. As a by product, we give an updated view of the classical tests providing, at the same time, some improvements in their formulation based on a recent reformulation of the discriminant of a cubic [20].

This paper is organized as follows. Section 2 summarizes the basic concepts and notations concerning conics and pencils of conics. Section 3 shows how to identify the root pattern of the characteristic polynomial of the pencil defined by two ellipses. Section 4 shows how to classify the elements of the pencil, and Section 5 discusses some properties of the pencils containing nested or disjoint ellipses. Then, in Section 6, it is shown that some relative locations between two ellipses can unambiguously be identified solely based on the studied root pattern. Nevertheless, a deeper analysis is required to complete the classification. To this end, the identification of double lines is discussed in Section 7; the identification of coincident ellipses, in Section 8; and the identification of real singular conics, in Section 9. As a result, the simple binary decision tree to identify the relative position between two ellipses is presented (see Fig. 4). Section 10 evaluates the computational cost of the presented approach and gives details on the supplementary downloadable software and the included examples. Finally, Section 11 summarizes the main results and gives prospects for further extensions.

2. Conics and pencils of conics

2.1. Conics

A real projective planar conic $\mathcal{C}$ is defined by a homogeneous polynomial $F(x, y, w)$ of degree 2 with real coefficients of the form:

$$F(x, y, w) = ax^2 + by^2 + 2cxy + 2dxw + 2eyw + fw^2 = 0, \tag{1}$$

which can be expressed in matrix form as

$$\mathbf{p}^T \mathbf{M}_{\mathcal{C}} \mathbf{p} = 0, \tag{2}$$

where $\mathbf{p} = (x\,y\,w)^T$ and

$$\mathbf{M}_{\mathcal{C}} = \begin{pmatrix} a & c & d \\ c & b & e \\ d & e & f \end{pmatrix}. \tag{3}$$

Given $\mathcal{C}$, $\mathbf{M}_{\mathcal{C}}$ is unique up to a constant non-zero scalar. The *inside* of $\mathcal{C}$ is the set of points of the projective plane satisfying

$$\det(\mathbf{M}_{\mathcal{C}}) F(x, y, w) > 0, \tag{4}$$

which is conserved under projective equivalence and is topologically isomorphic to an open disk.

The affine type of a real conic can be determined by reducing its equation to a canonical form, which essentially boils down to diagonalizing $\mathbf{M}_{\mathcal{C}}$ and computing the signs of its eigenvalues and of its upper left 2×2 sub-matrix eigenvalues [22, §7.7]. Nevertheless, for the purpose of this paper, we do not need the full affine classification of real conics, and we use a characterization (halfway between the real projective and the real affine classification) based on a minimum set of semi-algebraic conditions defining the nine types appearing in Fig. 2. To this end, we define the matrices

$$\mathbf{N}_{\mathcal{C}} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \ \mathbf{O}_{\mathcal{C}} = \begin{pmatrix} a & d \\ d & f \end{pmatrix}, \text{ and } \mathbf{S}_{\mathcal{C}} = \begin{pmatrix} b & e \\ e & f \end{pmatrix}. \tag{5}$$

Then, it can be proved that the considered nine types of conics can be discerned from the signs of $\det(\mathbf{M}_{\mathcal{C}})$, $\det(\mathbf{N}_{\mathcal{C}})$, $\det(\mathbf{O}_{\mathcal{C}})+\det(\mathbf{S}_{\mathcal{C}})$, and a, according to the decision tree depicted in Fig. 2. The deduction of this classification can be carried out in many different ways, but probably the simplest one, based on elementary arguments, can be found following the ideas of [23, pp. 80-81] (being aware that the classification provided in this reference concerning the degenerate conics is wrong).

2.2. Pencils of conics

Two conics, say $\mathcal{C}_1\colon \mathbf{p}^T \mathbf{M}_{\mathcal{C}_1} \mathbf{p} = 0$ and $\mathcal{C}_2\colon \mathbf{p}^T \mathbf{M}_{\mathcal{C}_2} \mathbf{p} = 0$, define the pencil of conics

$$\mathbf{p}^T (\mathbf{M}_{\mathcal{C}_2} + \lambda \mathbf{M}_{\mathcal{C}_1}) \mathbf{p} = 0, \ \lambda \in \mathbb{R} \cup \infty. \tag{6}$$

Any two conics in this pencil intersect in the same four points (real or complex) counted with multiplicity.

The values of λ for which a conic of this pencil is singular correspond to those in which

$$P(\lambda) = \det(\mathbf{M}_{\mathcal{C}_2} + \lambda \mathbf{M}_{\mathcal{C}_1}) = l_3\lambda^3 + 3l_2\lambda^2 + 3l_1\lambda + l_0 = 0 \tag{7}$$

where the coefficients l_i, $i = 0, 1, 2, 3$, can be expressed as [24, p. 274] [25, p. 191]:

$$l_3 = \begin{vmatrix} a_1 & c_1 & d_1 \\ c_1 & b_1 & e_1 \\ d_1 & e_1 & f_1 \end{vmatrix} = \det(\mathbf{M}_{\mathcal{C}_1}), \tag{8}$$

$$3l_2 = \begin{vmatrix} a_2 & c_1 & d_1 \\ c_2 & b_1 & e_1 \\ d_2 & e_1 & f_1 \end{vmatrix} + \begin{vmatrix} a_1 & c_2 & d_1 \\ c_1 & b_2 & e_1 \\ d_1 & e_2 & f_1 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & d_2 \\ c_1 & b_1 & e_2 \\ d_1 & e_1 & f_2 \end{vmatrix}, \tag{9}$$

$$3l_1 = \begin{vmatrix} a_1 & c_2 & d_2 \\ c_1 & b_2 & e_2 \\ d_1 & e_2 & f_2 \end{vmatrix} + \begin{vmatrix} a_2 & c_1 & d_2 \\ c_2 & b_1 & e_2 \\ d_2 & e_1 & f_2 \end{vmatrix} + \begin{vmatrix} a_2 & c_2 & d_1 \\ c_2 & b_2 & e_1 \\ d_2 & e_2 & f_1 \end{vmatrix}, \tag{10}$$

$$l_0 = \begin{vmatrix} a_2 & c_2 & d_2 \\ c_2 & b_2 & e_2 \\ d_2 & e_2 & f_2 \end{vmatrix} = \det(\mathbf{M}_{\mathcal{C}_2}). \tag{11}$$

The above polynomial, $P(\lambda)$, is known as the *characteristic polynomial* of the pencil, and λ, the *pencil parameter*.

3. The seven possible root patterns for $P(\lambda)$

Without loss of generality, we may assume that $\det(\mathbf{M}_{\mathcal{C}_i}) \leq 0$, otherwise $\mathbf{M}_{\mathcal{C}_i}$ can simply be substituted by $-\mathbf{M}_{\mathcal{C}_i}$ without modifying $\mathcal{C}_i$, $i = 1, 2$. Now, observe that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are not degenerate conics then $l_3 < 0$ and $l_0 < 0$. As a consequence,

$$P(-\infty) > 0, \ P(0) < 0, \text{ and } P(\infty) < 0. \tag{12}$$

Therefore, $P(\lambda) = 0$ will have at least one negative real root.

If λ_i, $1 \leq i \leq 3$ denote the three roots of $P(\lambda) = 0$, then Vieta's formulas read as follows:

$$\lambda_1\lambda_2\lambda_3 = -\frac{l_0}{l_3} < 0, \tag{13}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = -3\frac{l_2}{l_3}, \tag{14}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 3\frac{l_1}{l_3}. \tag{15}$$

Now, let us assume that the three roots are real and $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Since λ_1 is necessarily negative, according to (13) $\lambda_2\lambda_3 > 0$. As a consequence, λ_2 and

λ_3 are both positive or negative. If λ_2 and λ_3 are negative then, according to (14), $l_2 < 0$, and according to (15), $l_1 < 0$. A sharper result than the converse is also true: if λ_2 and λ_3 are positive then, $l_2 < 0$ or $l_1 < 0$. Indeed, let us suppose that $\lambda_2 > 0$, $\lambda_3 > 0$, and $l_2 < 0$, then, using (14), $\lambda_1 \leq -(\lambda_2+\lambda_3)$, and substituting in (15) gives $3\frac{l_1}{l_3} = \lambda_1(\lambda_2+\lambda_3) + \lambda_2\lambda_3 \leq -(\lambda_2+\lambda_3)^2 + \lambda_2\lambda_3 < 0$, that is, $l_1 > 0$. Hence, it is possible to conclude that λ_2 and λ_3 are positive if, and only if, $l_1 > 0$ or $l_2 > 0$. Otherwise, $l_1 < 0$ and $l_2 < 0$.

Important information on the roots of $P(\lambda)$ can also be derived from the sign of its discriminant, Δ_P. This discriminant can be expressed as [20]:

$$\Delta_P = \begin{vmatrix} 2\delta_1 & \delta_2 \\ \delta_2 & 2\delta_3 \end{vmatrix} \tag{16}$$

where

$$\delta_1 = \begin{vmatrix} l_3 & l_2 \\ l_2 & l_1 \end{vmatrix}, \quad \delta_2 = \begin{vmatrix} l_3 & l_1 \\ l_2 & l_0 \end{vmatrix}, \quad \text{and} \quad \delta_3 = \begin{vmatrix} l_2 & l_1 \\ l_1 & l_0 \end{vmatrix}. \tag{17}$$

In [20], it is also proved that $P(\lambda) = 0$ has a triple root if, and only if,

$$\delta_1 = \delta_2 = \delta_3 = 0. \tag{18}$$

It can be checked that $l_0\delta_1 = l_1\delta_2 - l_2\delta_3$. Then, since $l_0 \neq 0$, the three conditions in (18) can be reduced to $\delta_2 = \delta_3 = 0$.

The above analysis permits to classify the possible root patterns of $P(\lambda) = 0$ as shown in Fig. 3. These root patterns not only give information on the multiplicity of the singular conics in the pencil, but also valuable information on the relative position of $\mathcal{C}_1$ and $\mathcal{C}_2$ as explained next.

Finally, it is important observing that the triple root in root pattern 2 can be expressed as

$$\lambda_t = -\frac{l_2}{l_3}, \tag{19}$$

which equals the root of $P''(\lambda) = 0$, and the double and single roots in roots patterns 3a, 3b, and 4, as:

$$\lambda_d = -\frac{\delta_2}{2\delta_1}, \tag{20}$$

$$\lambda_s = -\frac{l_0\delta_2}{l_3\delta_3}, \tag{21}$$

respectively.

4. Classifying the elements of the pencil

Now, to classify the elements of the pencil, we have to study the sign of:

$$\det(\mathbf{N}_{\mathcal{C}_2} + \lambda\mathbf{N}_{\mathcal{C}_1}). \tag{22}$$

To this end, we can define the characteristic polynomial of the pencil of one-dimensional conics $\mathbf{N}_{\mathcal{C}_2} + \lambda \mathbf{N}_{\mathcal{C}_1}$:

$$Q(\lambda) = \det(\mathbf{N}_{\mathcal{C}_2} + \lambda \mathbf{N}_{\mathcal{C}_1}) = m_2\lambda^2 + 2m_1\lambda + m_0, \tag{23}$$

where, as above,

$$m_2 = \begin{vmatrix} a_1 & c_1 \\ c_1 & b_1 \end{vmatrix} = \det(\mathbf{N}_{\mathcal{C}_1}), \tag{24}$$

$$2m_1 = \begin{vmatrix} a_2 & c_1 \\ c_2 & b_1 \end{vmatrix} + \begin{vmatrix} a_1 & c_2 \\ c_1 & b_2 \end{vmatrix}, \tag{25}$$

$$m_0 = \begin{vmatrix} a_2 & c_2 \\ c_2 & b_2 \end{vmatrix} = \det(\mathbf{N}_{\mathcal{C}_2}). \tag{26}$$

Observe that, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are ellipses then, according to Fig. 2, $m_2 > 0$ and $m_0 > 0$. As a consequence,

$$Q(-\infty) > 0, \ Q(0) > 0, \ \text{and } Q(\infty) > 0. \tag{27}$$

Then, $(\lambda, Q(\lambda))$ represents a parabola opening upward.

The discriminant of (23) is given by

$$\Delta_Q = 4 \begin{vmatrix} m_1 & m_0 \\ m_2 & m_1 \end{vmatrix}. \tag{28}$$

Now observe that $\Delta_Q \geq 0$, otherwise there will be no conic in the pencil intersecting the line at infinity and this is impossible because, for each point of the line at infinity, there is a conic in the pencil passing through it. Therefore, the roots of $Q(\lambda) = 0$, say λ_{q_1} and λ_{q_2} with $\lambda_{q_1} \leq \lambda_{q_2}$, are always real (because $\Delta_Q \geq 0$) with the same sign (because $Q(0) > 0$).

The limit case $\Delta_Q = 0$ occurs if, and only if, the two ellipses intersect the line at infinity, $w = 0$, at the same two points (as, for instance, the case of two circles, no matter what their relative position is), and this condition is equivalent to having a singular conic in the pencil containing the line at infinity.

Notice that, according to Fig. 2 and the nature of $(\lambda, Q(\lambda))$, a singular conic of the pencil which is a pair of real (different or coincident) lines must have its pencil parameter in the closed interval $[\lambda_{q_1}, \lambda_{q_2}]$, whereas the parameter of a pair of imaginary lines belongs to $(-\infty, \lambda_{q_1}] \cup [\lambda_{q_2}, +\infty)$. Hence, due to the fact that $\lambda_1 < 0$, types 1, 2, 6, 9 and 10, where all singular conics are real, must fulfill $\lambda_{q_1} \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_{q_2} < 0$, and the roots of $P(\lambda) = 0$ cannot be arranged as in root pattern 4 nor 6.

5. Characterizing ellipses with nested or disjoint inside

The pencils of conics can be classified up to projective transformations as in [5, Ch. 4, §11] so that each class of pencils can have a distinguished pencil as representative. Simple expressions for these representatives can be obtained by taking a particular reference frame according to the degenerate conics in the pencil. This simplifies the study of those pencil characteristics that remain invariant under projective transformations because they only have to be proved for the representative of each class. This is the case of the existence of empty real conics (also named as imaginary ellipses), and the nested or disjoint relationship between insides. The study of these two characteristics is next carried out for a representative of the classes of pencils associated with the intersection sets of type $\mathrm{I_a}$, $\mathrm{II_a}$ and $\mathrm{III_a}$. It is worth to be noticed that the pencils associated with the intersection sets of type $\mathrm{I_a}$ and $\mathrm{III_a}$ are the only ones containing imaginary ellipses. The representative for each class will be denoted by $\mathcal{C}_\lambda = \mathcal{C}_2 + \lambda\mathcal{C}_1$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ have to be properly chosen for each class.

$\mathrm{I_a}$: In this case, the characteristic polynomial can be expressed as:

$$P(\lambda) = \det(\mathbf{M}_{\mathcal{C}_2} + \lambda\mathbf{M}_{\mathcal{C}_1}) = \left(1 + \frac{\lambda}{2}\right)\left(1 - \frac{\lambda}{2}\right). \tag{29}$$

Hence, the singular conics are a pair of real lines (for $\lambda_1 = \infty$), and two pairs of imaginary lines (for $\lambda_2 = 2$ and for $\lambda_3 = -2$). All the conics in the pencil with values for the pencil parameter in the open interval (λ_2, λ_3) are imaginary ellipses.

Now, if we take

$$\begin{aligned}\mathcal{C}_1 : F_1(x, y, w) = x^2 + y^2 + w^2 + \mu_1 xw = 0\\ \mathcal{C}_2 : F_2(x, y, w) = x^2 + y^2 + w^2 + \mu_2 xw = 0\end{aligned}$$

as the two ellipses defining the representative of the class, we have that

- $\{\det(\mathbf{M}_{\mathcal{C}_1})F_1(x, y, w) > 0\} \subseteq \{\det(\mathbf{M}_{\mathcal{C}_2})F_2(x, y, w) > 0\}$ if, and only if, either $\mu_1 > \mu_2 > 2$ or $\mu_1 < \mu_2 < -2$.
- $\{\det(\mathbf{M}_{\mathcal{C}_1})F_1(x, y, w) > 0\} \cap \{\det(\mathbf{M}_{\mathcal{C}_2})F_2(x, y, w) > 0\} = \emptyset$ if, and only if, either $\mu_1 > 2$ and $\mu_2 < -2$, or $\mu_2 > 2$ and $\mu_1 < -2$.

Summing up, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are inside one another, then the values μ_1 and μ_2 for the pencil parameter lie on the same connected component of $\{\mathbb{R} \cup \infty\} \backslash \{\lambda_1, \lambda_2, \lambda_3\}$. On the contrary, if their interiors are disjoint, these values for the pencil parameter lie on different connected components.

$\mathrm{II_a}$: In this case, the characteristic polynomial can be expressed as:

$$P(\lambda) = -\frac{\lambda^2}{4}. \tag{30}$$

Therefore, the singular conics of the pencil are a pair of real lines (for $\lambda_1 = \infty$), and two coincident pairs of imaginary lines (for $\lambda_2 = \lambda_3 = 0$). Now, if we take

$$\begin{aligned}\mathcal{C}_1 : F_1(x, y, w) = x^2 + y^2 + \mu_1 xw = 0\\ \mathcal{C}_2 : F_2(x, y, w) = x^2 + y^2 + \mu_2 xw = 0\end{aligned}$$

as the two ellipses defining the representative of the class, we have that

- $\{\det(\mathbf{M}_{\mathcal{C}_1})F_1(x,y,w) > 0\} \subseteq \{\det(\mathbf{M}_{\mathcal{C}_2})F_2(x,y,w) > 0\}$ if, and only if, either $\mu_2 > \mu_1 > 0$ or $\mu_2 < \mu_1 < 0$.
- $\{\det(\mathbf{M}_{\mathcal{C}_1})F_1(x,y,w) > 0\} \cap \{\det(\mathbf{M}_{\mathcal{C}_2})F_2(x,y,w) > 0\} = \emptyset$ if, and only if, either $\mu_1 > 0$ and $\mu_2 < 0$, or $\mu_1 > 0$ and $\mu_2 < 0$.

Summing up, as in the preceding case, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are inside one another, then the values μ_1 and μ_2 for the pencil parameter lie on the same connected component of $\{\mathbb{R} \cup \infty\} \backslash \{\lambda_1, \lambda_2, \lambda_3\}$. On the contrary, if their interiors are disjoint, they lie on different connected components.

$\mathrm{III_a}$: In this case, the characteristic polynomial can be expressed as:

$$P(\lambda) = \lambda. \tag{31}$$

As a consequence, the singular conics are two coincident of double lines (for $\lambda_1 = \lambda_2 = \infty$), and a pair of imaginary lines (for $\lambda_3 = 0$). All the conics in the pencil with values of the pencil parameter in the interval $(0, +\infty)$ are imaginary ellipses.

Now, if we take

$$\begin{aligned} \mathcal{C}_1 : F_1(x,y,w) = x^2 + y^2 + \mu_1 w^2 = 0 \\ \mathcal{C}_2 : F_2(x,y,w) = x^2 + y^2 + \mu_2 w^2 = 0 \end{aligned}$$

as the two ellipses defining the representative of the class, we have that

- $\{\det(\mathbf{M}_{\mathcal{C}_1})F_1(x,y,w) > 0\} \subseteq \{\det(\mathbf{M}_{\mathcal{C}_2})F_2(x,y,w) > 0\}$ if, and only if, $0 > \mu_1 > \mu_2$. Hence, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are inside one another, then the values μ_1 and μ_2 for the pencil parameter lie on the same connected component of $\{\mathbb{R} \cup \infty\} \backslash \{\lambda_1, \lambda_2, \lambda_3\}$.

6. Discrimination attained from the root pattern of $P(\lambda)$

Observe that the values of the pencil parameter for $\mathcal{C}_1$, $\lambda = \infty$, and for $\mathcal{C}_2$, $\lambda = 0$, lie on different connected components of $\{\mathbb{R} \cup \infty\} \backslash \{\lambda_1, \lambda_2, \lambda_3\}$ if, and only if, the roots of $P(\lambda) = 0$ are arranged as in root pattern 4 or 6.

Attending to the number and the nature of the singular conics for each type of positional relationship, it can be concluded that, if $\mathcal{C}_1$ and $\mathcal{C}_2$

1. are transversal at four points (type 1), then the roots of $P(\lambda) = 0$ are arranged as in root pattern 5 (since all 3 singular conics are real lines, the 3 different roots $\lambda_1, \lambda_2, \lambda_3$ are negative, which univocally matches with root pattern 5);
2. are transversal at two points (type 2), then the roots of $P(\lambda) = 0$ are arranged as in root pattern 1 (a single singular conic of multiplicity one univocally matches with root pattern 1);

3. are transversal at two points and tangent at another (type 6), or are internally tangent a two points (type 9), then the roots of $P(\lambda) = 0$ are arranged either as in pattern 3a or as in pattern 3b (the singular conic appearing twice has parameter λ_d and both λ_d and λ_s are negative since the two singular conics are real lines, which univocally matches with root pattern 3a or 3b);

4. have an osculating (type 10) or hyperosculating (type 11) contact, then the roots of $P(\lambda) = 0$ are arranged as in root pattern 2 (the singular conic appearing 3 times has parameter λ_t, which univocally matches with root pattern 2).

It has been proved in Section 5 that, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are inside one another, the values of the pencil parameter $\lambda = \infty$ and $\lambda = 0$ lie on the same connected component of $\{\mathbb{R} \cup \infty\} \backslash \{\lambda_1, \lambda_2, \lambda_3\}$. Then, in this case, only pattern 2, 3a, 3b or 5, is possible. If, the interiors of $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint, these values for the pencil parameter lie on different connected components, which means that only pattern 4, or 6, is possible. Therefore, we conclude that, if $\mathcal{C}_1$ and $\mathcal{C}_2$

- are separated (type 3), then the roots of $P(\lambda) = 0$ are arranged as in root pattern 6 [1];
- are externally tangent (type 7), then the roots of $P(\lambda) = 0$ are arranged as in root pattern 4;
- are internally tangent at one point (type 8), then the roots of $P(\lambda) = 0$ are arranged either as in pattern 3a or as in pattern 3b;
- one contained in the other, then the roots of $P(\lambda) = 0$ are arranged either as in pattern 5 (in case of type 4) or as in patterns 3a or 3b (in case of type 5).

The above results are summarized in the rightmost column of Table 1. Thus, it is possible to perform a certain classification of the relative location of two ellipses attending solely to the root pattern of the characteristic polynomial of the pencil they define. Unfortunately, this root pattern does not give enough information to complete the classification, it is also necessary to go deeper into the nature of the singular and imaginary (if any) conics in the pencil.

7. Conditions for the pencil to contain a double line

According to Fig. 2, if the system of equations

$$\left.\begin{array}{l} P(\lambda) = \det(\mathbf{M}_{\mathcal{C}_2} + \lambda \mathbf{M}_{\mathcal{C}_1}) = 0 \\ Q(\lambda) = \det(\mathbf{N}_{\mathcal{C}_2} + \lambda \mathbf{N}_{\mathcal{C}_1}) = 0 \\ R(\lambda) = \det(\mathbf{O}_{\mathcal{C}_2} + \lambda \mathbf{O}_{\mathcal{C}_1}) + \det(\mathbf{S}_{\mathcal{C}_2} + \lambda \mathbf{S}_{\mathcal{C}_1}) = 0 \end{array}\right\}$$

has a real root for λ, the pencil contains a double line. This necessary and sufficient condition readily permits to discern between contacts of type 10 and type 11, despite they have the same root pattern for $P(\lambda) = 0$, because the latter type contains a double line (repeated three times) in its pencil. Since the triple root in root pattern 2, according to (19), is $\lambda_t = -l_2/l_3$, the contact between the two ellipses is of type 11 if $Q(\lambda_t) = R(\lambda_t) = 0$, and type 10, otherwise.

This reasoning also permits to discern between the contacts with root patterns 3a and 3b into the group formed by contacts of type 8 and type 6, and the group formed by type 9 and type 5, because the contacts in the latter group contain a double line (appearing twice) in their pencils. In this case, since the double root in root pattern 3a and 3b is, according to (20), $\lambda_d = -\delta_2/2\delta_1$, the contact between the two ellipses is of type 9 or type 5 if $Q(\lambda_d) = R(\lambda_d) = 0$, and of type 8 or type 6, otherwise.

8. Conditions for coincident ellipses

Despite the case of coincident ellipses could be readily identified at the very beginning by checking if their matrices are proportional

$$\mathbf{M}_{\mathcal{C}_2} = \lambda_0 \mathbf{M}_{\mathcal{C}_1}, \ \lambda_0 > 0, \tag{32}$$

we will characterize this circumstance in a different way by adding a single algebraic condition in our binary decision tree.

For coincident ellipses, according to (32), we have that

$$\begin{aligned}
P(\lambda) &= \det(\mathbf{M}_{\mathcal{C}_1})(\lambda + \lambda_0)^3 \,, \\
Q(\lambda) &= \det(\mathbf{N}_{\mathcal{C}_1})(\lambda + \lambda_0)^2 \,, \\
R(\lambda) &= \left(\det(\mathbf{O}_{\mathcal{C}_1}) + \det(\mathbf{S}_{\mathcal{C}_1})\right)(\lambda + \lambda_0)^2 \,.
\end{aligned}$$

Hence $P(\lambda)$ has a triple root $\lambda_t = -\lambda_0$, which is also the double root of $Q(\lambda)$ and $R(\lambda)$.

Assuming that the roots of $P(\lambda)$ follow root pattern 2 and that $Q(\lambda_t) = R(\lambda_t)$, then $\Delta_Q = 0$ characterizes the case of coincident ellipses. Indeed, if $\Delta_Q = 0$, but the ellipses are not coincident, then their relative position must be of type 11. Then imposing $\Delta_Q = 0$ implies, as argued in Section 4, that a singular conic in the pencil contains the line at infinity, which makes no sense since the only singular conic in type 11 is a double line tangent to both ellipses in their hyperosculating contact point.

9. Conditions for the singular conics in the pencil to be real

As pointed out at the end of Section 4, verifying if a singular conic in the pencil is real reduces to verify that the corresponding root of $P(\lambda)$ is within the range defined by the roots of $Q(\lambda)$. However, special attention must be paid to the case in which a root of $P(\lambda)$ is also a root of $Q(\lambda)$ because, according to Fig. 2, this can happen for both real or imaginary lines, as long as they are parallel.

Throughout this section we will assume that the roots of $P(\lambda)$ follow root pattern 5, 3a or 3b, since the types with the other root patterns have already been fully characterized. According to Table 1, we have to disambiguate between type 5 and type 9, between type 6 and type 8, and between type 1 and type 4. This disambiguation is possible by realizing that the pencils for type 4, type 5 and type 8 contain at least one pair of non-parallel imaginary lines. In other words, in these cases at least one of the roots of $P(\lambda)$ makes $Q(\lambda)$ positive. Moreover, it is important to realize the following facts about these three types:

F1: In type 8, the pair of imaginary lines is the element of the pencil for λ_d, and the imaginary lines intersect in the common tangency point of the two ellipses, which prevents the imaginary lines from being parallel. Hence $Q(\lambda_d) > 0$.

F2: In type 5, the pair of imaginary lines is the element of the pencil for λ_s. Moreover, the conics for $\lambda \in (\lambda_s, \lambda_d)$ in case of root pattern 3a, or for $\lambda \in (\lambda_d, \lambda_s)$ in case of root pattern 3b, consists exclusively of imaginary ellipses. Now, since $Q(\lambda_d) = 0$ and $Q(0) = 0$, it follows that $Q(\lambda_s) > 0$ in both cases.

F3: In type 4, the two pairs of imaginary lines are the elements of the pencil either

- for λ_1 and λ_2 (in this case, the conics for $\lambda \in (\lambda_1, \lambda_2)$ consists exclusively of imaginary ellipses and, as a consequence, $Q(\lambda_1) > 0$); or
- for λ_2 and λ_3 (in this case, the conics for $\lambda \in (\lambda_2, \lambda_3)$ consists exclusively of imaginary ellipses and, as a consequence, $Q(\lambda_3) > 0$).

From F1 and F2, the disambiguation between type 5 and type 9, and between type 6 and type 8, becomes trivial because in these cases the roots of $P(\lambda)$ follow root pattern 3a or 3b and, according to (20) and (21), we have explicit simple algebraic expressions for λ_s and λ_d.

Disambiguating between type 1 and type 4 can also be performed by verifying if all singular conics in the pencil are real. Nevertheless, in this case the analysis is trickier that above. Since these two types follow root pattern 5, all singular conics in the pencil are real if, and only if, the roots of $P(\lambda)$, λ_1, λ_2, and λ_3, are within the range defined by the roots of $Q(\lambda)$, $[\lambda_{q_1}, \lambda_{q_2}]$, as depicted in Fig. 5. Indeed, deserving solely attention the case in which $\lambda_1 = \lambda_{q_1}$ and $\lambda_3 = \lambda_{q_2}$, the result follows from F3.

At this point it is important to observe that $\lambda_2 \in (\lambda_{q_1}, \lambda_{q_2})$ if, and only if, $[\lambda_1, \lambda_3] \subset [\lambda_{q_1}, \lambda_{q_2}]$, again due to F3. Thus, we have to simply concentrate our efforts in verifying if the singular conic in the pencil for λ_2 is real.

Now, observe that λ_2 is necessarily within the range of the roots of $P'(\lambda)$, $(\lambda_{p_1}, \lambda_{p_2})$, which is a subrange of (λ_1, λ_3) (see again Fig. 5). Then, we conclude that $\lambda_2 \in (\lambda_{q_1}, \lambda_{q_2})$ if, and only if, $(\lambda_{p_1}, \lambda_{p_2}) \subset (\lambda_{q_1}, \lambda_{q_2})$. To verify this latter condition, let us consider the intersections of an element of the pencil defined by $P'(\lambda) + \alpha Q(\lambda) = 0$ and the abscissa. The two intersections, say x and x^*

(which coincide with λ_{p_1} and λ_{p_2} for $\alpha = 0$, and with λ_{q_1} and λ_{q_2} for $\alpha = \infty$), are the roots of

$$P'(\lambda) + \alpha Q(\lambda) = (l_3 + \alpha\, m_2)\lambda^2 + 2(l_2 + \alpha\, m_1)\lambda + (l_1 + \alpha\, m_0) = 0. \tag{33}$$

Vieta's formulas lead, in this case, to

$$x + x^* = -2\frac{l_2 + \alpha\, m_1}{l_3 + \alpha\, m_2}, \tag{34}$$

$$xx^* = \frac{l_1 + \alpha\, m_0}{l_3 + \alpha\, m_2}. \tag{35}$$

Eliminating α from the above two equations yields

$$s_2 xx^* + \frac{1}{2}s_1(x + x^*) + s_0 = 0, \tag{36}$$

where

$$s_2 = \begin{vmatrix} l_2 & l_3 \\ m_1 & m_2 \end{vmatrix}, \; s_1 = \begin{vmatrix} l_1 & l_3 \\ m_0 & m_2 \end{vmatrix}, \; s_0 = \begin{vmatrix} l_1 & l_2 \\ m_0 & m_1 \end{vmatrix}. \tag{37}$$

Observe that equation (36) defines an involution on a line [26, §15], provided the line skips the base points of the pencil of conics. This is consequence of the Desargues' involution theorem that states that a line intersects the elements of a pencil of conics in reciprocal pairs of an involution on the line.

Involutions are distinguished by their fixed points, that is, by points reciprocal to themselves, that is, points satisfying $x^* = x$. Then, the fixed points of the involution (36) are given by the roots of

$$W(x) = s_2\, x^2 + s_1\, x + s_0 = 0. \tag{38}$$

Then, according as

$$\Delta_W = \begin{vmatrix} s_1 & 2s_2 \\ 2s_0 & s_1 \end{vmatrix} \gtreqless 0, \tag{39}$$

the fixed points are real, real and coincident or imaginary. The equality $\Delta_W = 0$ means that the abscissa goes through a base point of the pencil, and this cannot occur in an involution. In the former and latter cases, the involution is then called hyperbolic or elliptic, respectively. Alternatively, an involution can be classified as hyperbolic or elliptic by simply checking if the cross ratio of two different pairs of reciprocal points is positive [22, 5.6.12]. As a consequence, the generated involution is hyperbolic if, and only if, $(\lambda_{p_1}, \lambda_{p_2}) \subset (\lambda_{q_1}, \lambda_{q_2})$, $(\lambda_{p_1}, \lambda_{p_2}) \supset (\lambda_{q_1}, \lambda_{q_2})$ or $(\lambda_{p_1}, \lambda_{p_2}) \cap (\lambda_{q_1}, \lambda_{q_2}) = \emptyset$. The second case can never occur, neither for type 1 nor type 4, because it would imply that the conics in the pencil for λ_1 and λ_2 would be two pairs of imaginary lines, in contradiction with F3. To discard the third case, we can check if $Q(\lambda_t) < 0$, which, if satisfied, gives $\lambda_t \in (\lambda_{p_1}, \lambda_{p_2}) \cap (\lambda_{q_1}, \lambda_{q_2})$. Thus, all the degenerate conics in a pencil with root pattern 5 are real if, and only if, $Q(\lambda_t) < 0$ and $\Delta_W > 0$. This result finally permits to complete the binary decision tree for the positional relationship between two ellipses depicted in Fig. 4.

Notice that a pencil with root pattern 5 is of type 4 if, and only if, $Q(\lambda_t) > 0$ or $\Delta_W < 0$, which are open conditions. At first glance, according to the binary decision tree, type 4 is characterized by the condition $Q(\lambda_t) \geq 0$ or $\Delta_W \leq 0$. Nevertheless, in type 4, first the equality $\Delta_W = 0$ cannot occur, and second the equality $Q(\lambda_t) = 0$ implies the strict inequality $\Delta_W < 0$. Indeed, $\Delta_W = 0$ implies the existence of a base point of the pencil of conics $P'(\lambda) + \alpha Q(\lambda) = 0$ on the abscissa, but the restrictions imposed by F3 to type 4 imply that $(\lambda_{p_1}, \lambda_{p_2}) \supset (\lambda_{q_1}, \lambda_{q_2})$, which is absurd, as noticed above. Now, if $Q(\lambda_t) = 0$, then $\lambda_t \in (\lambda_{q_1}, \lambda_{q_2})$ and, together with $\lambda_t \in (\lambda_{p_1}, \lambda_{p_2})$, in accordance with the above reasoning, implies that the cross ratio of the two different pairs of reciprocal points $(\lambda_{q_1}, \lambda_{q_2}, \lambda_{p_1}, \lambda_{p_2})$ is negative, that is, $\Delta_W < 0$.

10. Computational cost and implementation

In the presented decision tree, the most costly operation is performed at the root of the tree (*i.e.*, the computation of the sign of Δ_P). This entails computing l_i, $i = 0, \ldots, 3$, which are cubic expressions in the ellipses' coefficients, then computing δ_i, $i = 1, \ldots, 3$, which are quadratic in l_i, and finally computing Δ_P which is also quadratic in δ_i. In other words, if these dependencies are expanded, Δ_P can be directly expressed as a polynomial of order 12 in terms of the ellipses' coefficients.

Contrary to what happens with previous approaches in which all operations essentially reduce to the evaluation of polynomials [16, 19], the one described here also involves rational algebraic expressions: those relative to λ_t (19), λ_d (20), and λ_s (21). Actually, their introduction is precisely the point that makes the presented algorithm comparatively so compact. Thus, comparing the presented approach with the mentioned ones in terms of polynomial degrees does not seem appropriate. As an alternative, the number of additions, multiplications, and divisions, required for the computation of the different operations is compiled in Table 2. The figures given in this table correspond to the brute-force evaluation of the corresponding formulas without accounting for repeated operations. Moreover, the coefficients of $R(\lambda)$ are denoted by t_i, with $i = 0, 1, 2$.

The simplicity of the proposed algorithm becomes apparent when reading its implementation. The supplementary downloadable material contains the MATLAB® function `RelativePosition.m` that implements the decision tree in Fig. 4. This function takes as input the matrices defining the two ellipses and returns their spatial relationship. The script `Main.m` contains one example of each possible spatial relationship.

At this point, it is important to observe that only the positional relationships of type 1, 2, 3 and 4, are stable with respect to small variations in the coefficients defining the ellipses. Indeed, according to what has been argued in Section 3 and 9, these four types are characterized by the following open conditions:

- Type 1: $\{\Delta_P > 0,\ l_1 < 0,\ l_2 < 0,\ Q(\lambda_t) < 0,\ \Delta_W > 0\}$;
- Type 2: $\{\Delta_P < 0\}$;

- Type 3: $\{\Delta_P < 0, l_1 > 0\}$ or $\{\Delta_P < 0, l_2 > 0\}$;
- Type 4: $\{\Delta_P > 0, l_1 < 0, l_2 < 0, Q(\lambda_t) > 0\}$ or $\{\Delta_P > 0, l_1 < 0, l_2 < 0, \Delta_W < 0\}$.

The remaining types include in their characterization the equality $\Delta_P = 0$. This means that, if we want to correctly identify unstable positional relationships in general, it would be necessary to implement the algorithm in exact rational arithmetics. In the current implementation, all calculations are done in floating-point arithmetics with certain margins in all comparisons to avoid missing some spatial relationships, but this is something that has to be adapted to each particular need.

In those cases in which the considered ellipses depend on some parameters, the problem usually consists in obtaining the value of these parameters for which some particular relationships are attained. In these cases, the presented decision tree can be used to obtain the set of algebraic equations that have to be solved simultaneously.

Finally, notice that the conditions describing type 5 are limit conditions for those describing type 4. Indeed, according to the binary decision tree in Fig. 4, and the results of Section 5 and 9, the conditions describing type 5 imply $\{\Delta_P = 0, l_1 < 0, l_2 < 0, Q(\lambda_t) > 0, \Delta_W = 0\}$, and hence they are adjacent to those of type 4. This has to be taken into account when dealing with time-varying ellipses.

11. Conclusion

We have presented a binary decision tree to classify the spatial relationship between two ellipses. The number of required operations is reduced compared to previous approaches, thanks to the introduction of explicit rational expressions for some polynomial roots, so that it can be adapted to any practical situation in which some particular positional relationships have to be discerned in a computationally efficient way.

The involved mathematical expressions at each level of the decision tree are simple enough algebraic or semi-algebraic conditions as to allow the possibility of analyzing the spatial relationship between continuously time-varying ellipses without relying on sampling. This is certainly a point that deserves further attention.

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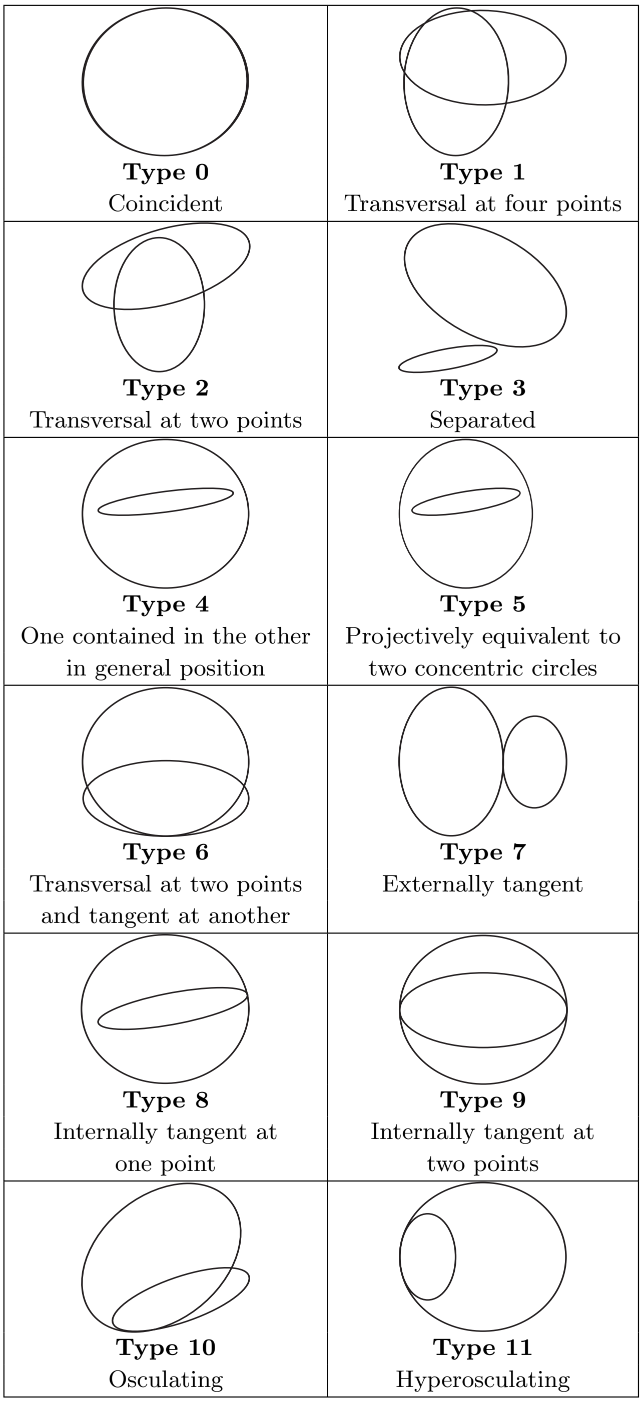

Figure 1: The considered twelve types of positional relationships between two ellipses.

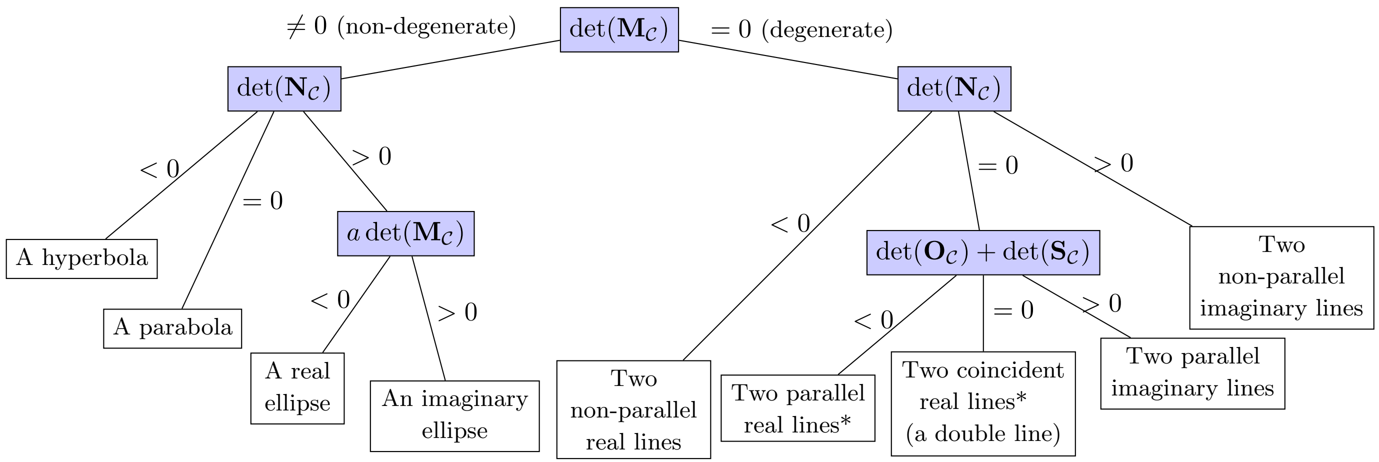

Figure 2: Classification of a real conic section. The cases with an asterisk (*) may include the line at infinity.

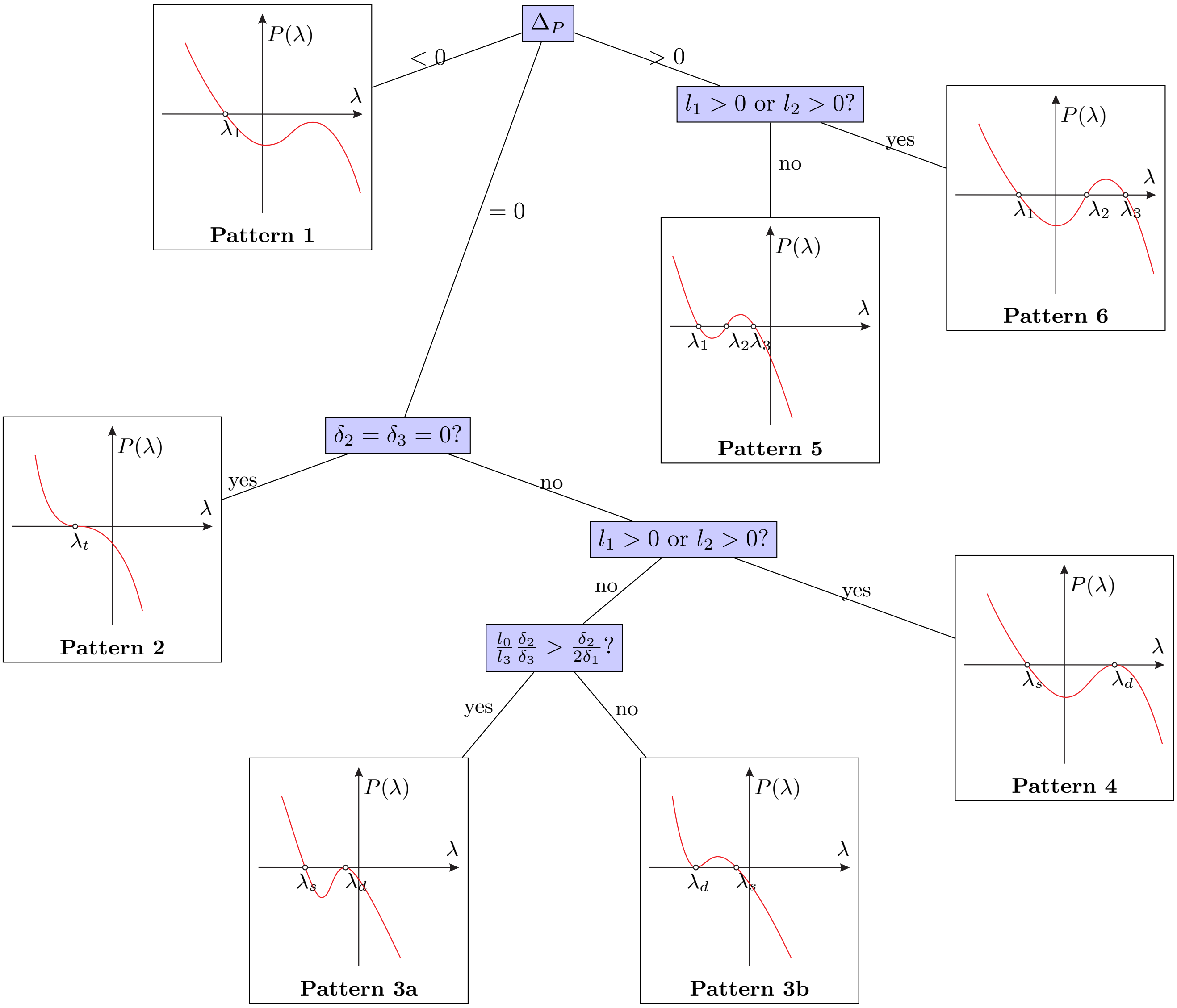

Figure 3: The seven possible root patterns for $P(\lambda) = 0$.

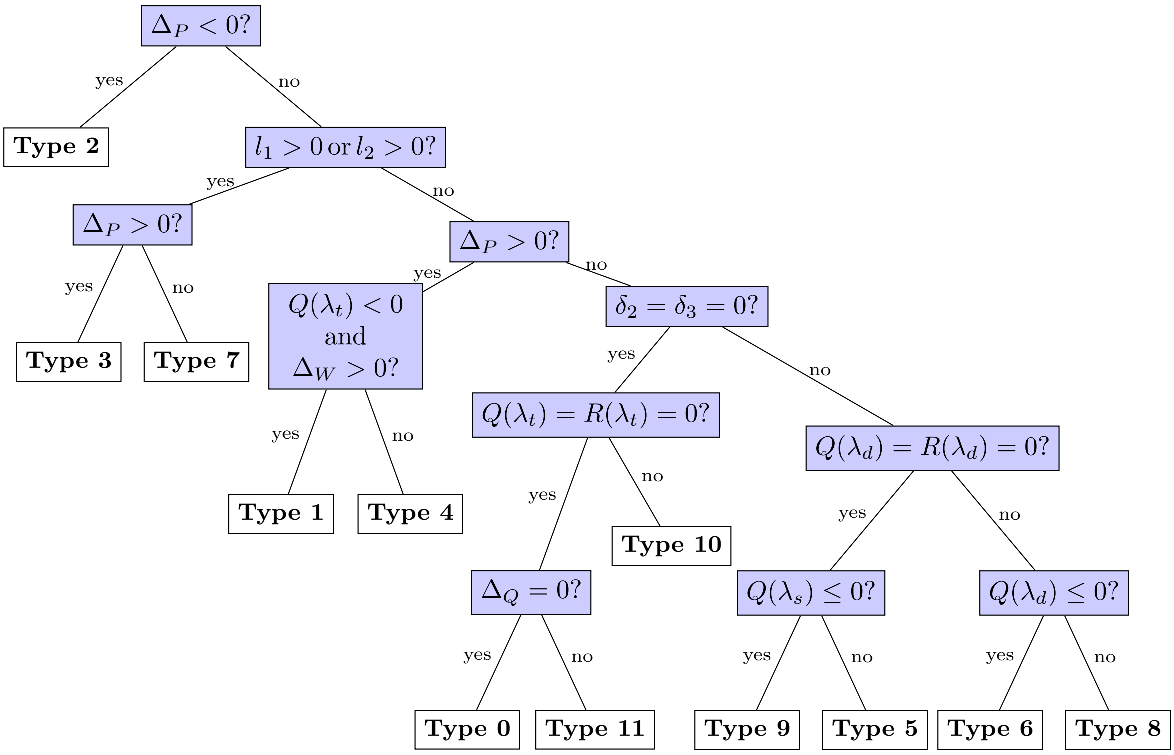

Figure 4: Binary decision tree to classify the relative position of two ellipses in one of the twelve types appearing in Fig. 1.

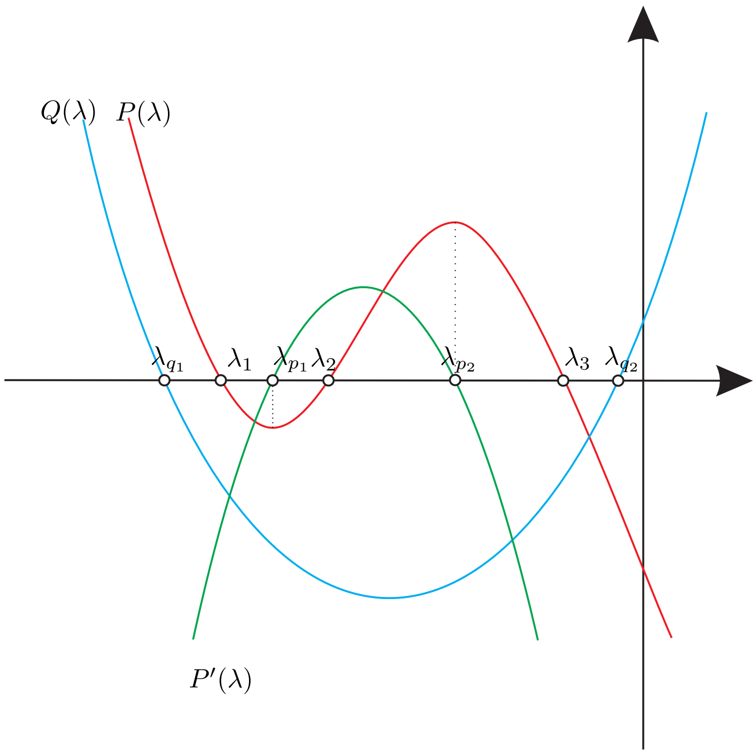

Figure 5: Arrangement of plots of $P(\lambda)$, $P'(\lambda)$, and $Q(\lambda)$ in the case that all singular conics in the pencil are real.

Table 1: Some characteristics of the considered positional relationships

Type	Real base points	Intersection set	Singular conics in the pencil	Root pattern of $P(\lambda) = 0$
1	4 simple	$\mathrm{I_c}$	3 pairs of real lines	5
2	2 simple	$\mathrm{I_b}$	1 pair of real lines	1
3	None	$\mathrm{I_a}$	1 pair of real lines 2 pairs of imaginary lines	6
4	None	$\mathrm{I_a}$	1 pair of real lines 2 pairs of imaginary lines	5
5	None	$\mathrm{III_a}$	1 double real line (twice) 1 pair of imaginary lines	3a, 3b
6	2 simple 1 double	$\mathrm{II_b}$	2 pairs of real lines (1 of them twice)	3a, 3b
7	1 double	$\mathrm{II_a}$	1 pair of real lines 1 pair of imaginary lines (twice)	4
8	1 double	$\mathrm{II_a}$	1 pair of real lines 1 pair of imaginary lines (twice)	3a, 3b
9	2 double	$\mathrm{III_b}$	1 pair of real lines 1 double real line (twice)	3a, 3b
10	1 simple 1 triple	IV	1 pair of real lines (3 times)	2
11	1 quadruple	V	1 double real line (3 times)	2

Table 2: Computational cost in terms of arithmetic operations

Variable	Equation	Additions	Multiplications	Divisions
l_3	(8)	5	9	0
l_2	(9)	17	27	1
l_1	(10)	17	27	1
l_0	(11)	5	9	0
m_2	(24)	1	2	0
m_1	(25)	3	4	1
m_0	(26)	1	2	0
t_2		3	4	0
t_1		7	8	2
t_0		3	4	0
δ_3	(17)	1	2	0
δ_2	(17)	1	2	0
δ_1	(17)	1	2	0
Δ_P	(16)	1	4	0
Δ_Q	(28)	1	2	0
λ_t	(19)	0	0	1
λ_d	(20)	0	1	1
λ_s	(21)	0	2	1
$Q(\cdot)$	(23)	2	5	0
$R(\cdot)$		2	5	0