KINESTATIC ANALYSIS OF SERIAL AND PARALLEL ROBOT MANIPULATORS USING GRASSMANN-CAYLEY ALGEBRA

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Abstract. In this paper the statics and the instantaneous kinematics of serial and parallel robot manipulators are studied. A projective interpretation of the concepts of twist, wrench, twist space and wrench space – based on the concept of extensor – is presented and a description of the dualistic relation between twist and wrench spaces of serial and parallel robot manipulators is given in terms of the Grassmann-Cayley algebra. The importance of this algebra is that its join and meet operators are very effective tools for joining and intersecting the linear subspaces involved in the kinestatic analysis of manipulators when they are represented by extensors.

1. Introduction

Any instantaneous motion of a rigid body can be described as a twist on a screw and any set of forces and moments that act on a rigid body can be described by means of a wrench on a screw.

Any possible twist of a rigid body partially constrained by a wrench is characterized by the *reciprocity condition*, that is, the work generated by the twist against the wrench must be zero and the term *kinestatics* refers to this dualistic relation between the statics and the instantaneous kinematics of rigid bodies. Mathematical frameworks of both differential and projective geometry have been used to describe this condition, also called duality, and relevant references on this subject are Ball (1900), Hunt (1990), Duffy (1990), Bruyninckx and De Schutter (1998) and Bruyninckx (1999).

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In this paper we gain a deeper understanding of the projective interpretation of kinestatics for serial and parallel robot manipulators using the *Grassmann-Cayley algebra* extending the work of White (1994) where the possible applications of this algebra to robotics are limited to the instantaneous kinematics of manipulators.

The works of Doubilet, Rota and Stein (1974) and Barnabei, Brini and Rota (1985) are comprehensive surveys on the properties of the Grassmann-Cayley algebra while White (1995) and White (1997) emphasize the concrete approach to this algebra and give more details on the connection to robotics, in particular, to the analysis of singular configurations of manipulators.

Sect. 2 reviews the most relevant properties of a version of the Grassmann-Cayley algebra based on the Plücker coordinates of linear subspaces of a given vector space and those of their duals, also called dual Plücker coordinates.

In Sect. 3 the representation of generalized forces and velocities in a projective setting is explained. It is based on the concept of extensor that corresponds, in the coordinated version, just to the vector of Plücker coordinates of a linear subspace. Since any element of a vector subspace is reciprocal to any element of its dual, twist and wrench spaces of partially constrained rigid bodies can be described by means of the Plücker coordinates and the dual Plücker coordinates, respectively.

Moreover, by using the join and meet operators of the Grassmann-Cayley algebra, which correspond to the sum and intersection of linear subspaces, respectively, the twist and the wrench spaces resulting from serial and parallel combinations of kinematic constraints can be easily expressed. This is discussed in Sect. 4.

The importance of the Grassmann-Cayley algebra is that it has an explicit formula for the meet operator and that it contains intrinsically a useful dualistic property: the meet and the join operators can be interchanged in a given expression if we change the arguments with their duals. For this reason it is also called *double algebra.*

Finally, Sect. 5 contains an example in which the Grassmann-Cayley algebra is used in the kinestatic analysis of a parallel manipulator.

2. The Grassmann-Cayley Algebra

In this section we report the most relevant properties of a version of the Grassmann-Cayley algebra that involves Plücker coordinates. Further details can be found in White (1994).

We start from some basic concept about projective space and homogeneous coordinates. If $\mathbf{x} = (x_1, x_2, \dots, x_m)^t$ is a point in $\Re^m$ given in terms of *Cartesian coordinates*, the vector $x = (x_1, x_2, \cdots, x_m, 1)^t$ is defined to be its *homogeneous coordinate* vector.

If we allow points with the last coordinate 0 for representing *projective points* at infinity, the standard m-dimensional *projective space* that includes $\Re^m$ is obtained. So, we represent $\Re^m$ with points at infinity by a $(m+1)$-dimensional vector space V.

2.1. PLÜCKER COORDINATES

Let U be a k-dimensional subspace of the $(m+1)$-dimensional vector space V, and $\{u_1\ u_2\ \ldots\ u_k\}$ a basis of it. When these vectors are arranged as rows of a matrix, we obtain:

$$\begin{pmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,m} & 1 \\ u_{2,1} & u_{2,2} & \cdots & u_{2,m} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{k,1} & u_{k,2} & \cdots & u_{k,m} & 1 \end{pmatrix}.$$

The $(j_1, j_2, \cdots, j_k)$-th *Plücker coordinate* of the subspace U, denoted by $P_{j_1, j_2, \cdots, j_k}$, is the $k \times k$ determinant formed by the k columns of the above matrix with indices $j_1, j_2, \cdots, j_k$. Since we have a Plücker coordinate for each combination of the k columns, the total number of Plücker coordinates is $\binom{m+1}{k}$. The *Plücker coordinate vector* of the subspace U is the vector P_U that contains in some predetermined order its Plücker coordinates. U uniquely determines P_U up to a scalar multiple.

The Plücker coordinate vector of the line Λ passing through points a and b, represented in homogeneous coordinates by the rows of

$$\begin{pmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \end{pmatrix},$$

is defined as:

$$P_\Lambda = (-P_{1,4},\ -P_{2,4},\ -P_{3,4},\ P_{2,3},\ -P_{1,3},\ P_{1,2})^t =$$
$$(b_1 - a_1, b_2 - a_2, b_3 - a_3, a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)^t = (\mathbf{S}, \mathbf{r} \times \mathbf{S})^t$$

where $\mathbf{S} = \mathbf{b} - \mathbf{a}$, $(\mathbf{r} \times \mathbf{S})$ represents the moment of Λ with respect to the origin, and $\mathbf{r}$ is any point on the line. The vector P_Λ is determined by Λ up to a scalar multiple and, since it will be used to represent rotations, it is convenient to normalize it such that $\mathbf{S} \cdot \mathbf{S} = 1$.

The point at infinity on Λ has homogeneous coordinates $(b_1 - a_1, b_2 - a_2, b_3 - a_3, 0)^t$ and can be thought as infinitely far away in the direction given by $\mathbf{S}$.

A line at infinity is determined by two distinct points, c and d, at infinity represented by the rows of

$$\begin{pmatrix} c_1 & c_2 & c_3 & 0 \\ d_1 & d_2 & d_3 & 0 \end{pmatrix}$$

whose Plücker coordinate vector is

$$P_\Lambda = (0,0,0,c_2d_3 - c_3d_2, c_3d_1 - c_1d_3, c_1d_2 - c_2d_1)^t = (0,0,0,\mathbf{r}\times\mathbf{S})^t.$$

Since lines at infinity will be used to represent translations, it is convenient to normalize this vector in such a way that $(\mathbf{r}\times\mathbf{S})\cdot(\mathbf{r}\times\mathbf{S}) = 1$.

The Plücker coordinate vector of the plane Π, determined by three finite points whose homogeneous coordinates are the rows of

$$\begin{pmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ p_1 & p_2 & p_3 & 1 \end{pmatrix},$$

is defined as

$$P_\Pi = (P_{2,3,4},\ -P_{1,3,4},\ P_{1,2,4},\ P_{1,2,3})^t = (\mathbf{N}, \mathbf{r}\cdot\mathbf{N})^t,$$

where $\mathbf{N}$ is a normal vector to Π and $\mathbf{r}$ is the position vector of any point on Π.

2.2. DUAL PLÜCKER COORDINATES

Let U be a k-dimensional subspace of the $(m+1)$-dimensional vector space V and $\{u_1,\ u_2,\ \dots\ ,u_k\}$ a basis of it. We can build up the dual of the vector subspace U as follows. Consider the linear system of equations

$$\sum_{j=0}^{m+1} x_j u_{h,j} = 0,\ h = 1,2,\dots,k.$$

Since its matrix has rank k its solution space is a $(m+1-k)$-dimensional vector space U^*. Let $\{w_1,\ w_2,\ \dots\ ,\ w_{m+1-k}\}$ be a basis of U^* we have the following relationships between the basis vectors of U and U^*

$$\sum_{j=0}^{m+1} w_{i,j} u_{h,j} = 0,\ i = 1,2,\dots,(m+1-k),\ h = 1,2,\dots,k \qquad (1)$$

by which we can compute the basis of U^* from the basis of U up to a scalar multiple. The vector spaces U and U^* are said to be *dual spaces*. Thus, the *dual Plücker coordinate vector* of U is defined to be the vector that contains the Plücker coordinates of U^* called *dual Plücker coordinates* of U.

It can be proved that the number of dual Plücker coordinates equals the number of Plücker coordinates, the only difference between them being their ordering and some sign changes. For example, the Plücker and dual Plücker coordinate vectors of lines in $\Re^3$ have the first three and the last three elements interchanged.

We also have

$$\sum_{j=0}^{m+1} w_{i,j}x_j = 0, \; i = 1, 2, \ldots, (m+1-k).$$

Since the w_i can thought as hyperplanes each containing U, these equations express the fact that the same subspace U can be represented as the subspace spanned by the basis vectors u_i or as the intersection of the hyperplanes w_i. A line in $\Re^3$, for example, can be constructed in two ways as the join of two points or as the meet of two planes. Traditionally, the Plücker coordinates of lines are called *ray coordinates* while the dual Plücker coordinates are called *axis coordinates* of the line.

If we replace vectors of V by hyperplanes we obtain its dual vector space V^*.

2.3. THE JOIN AND MEET OPERATORS

Let V be a n-dimensional vector space over the field $\Re$, U a k-dimensional subspace of V, and $\{u_1, u_2, \ldots, u_k\}$ a basis of it. Let also P be the Plücker coordinate vector of U, that is, a vector in the $\binom{n}{k}$-dimensional vector space $V^{(k)}$. The vector P is called a *k-extensor* which is denoted by

$$P = \vee(u_1\, u_2 \cdots u_k) = u_1 \vee u_2 \vee \cdots \vee u_k.$$

The subspace U, also denoted by $\bar{P}$, is defined as the *support* of P and the scalar k is defined to be the *step* of the extensor. If $n = 4$, that is, if we are using homogeneous coordinates in $\Re^3$, the support of a 2-extensor is a line and the support of a 3-extensor is a plane. Two k-extensors are equal up to a scalar multiple if, and only if, their have the same support.

Let $A = a_1 \vee a_2 \vee \cdots \vee a_k$ and $B = b_1 \vee b_2 \vee \cdots \vee b_j$ be two extensors. The *join* of A and B is defined as the $(j+k)$-extensor

$$A \vee B = a_1 \vee a_2 \vee \cdots \vee a_k \vee b_1 \vee b_2 \vee \cdots \vee b_j.$$

If the vectors $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_j\}$ are linearly dependent, then $A \vee B = 0$. If they are independent,

$$\overline{A \vee B} = \bar{A} + \bar{B} = \mathrm{span}(\bar{A} \cup \bar{B}). \tag{2}$$

This means that the join of two extensors represent the operation of joining the associated vector subspaces.

Now, another operation that plays a similar role for the intersection of subspaces is defined. Let A and B be the above two extensors with $(k+j) \geq n$. The *meet* of A and B is defined as:

$$A \wedge B =$$

$$\sum_{\sigma} sgn(\sigma)[a_{\sigma(1)}, \ldots, a_{\sigma(n-j)}, b_1, \ldots, b_j]\; a_{\sigma(n-j+1)} \vee a_{\sigma(n-j+2)} \vee \cdots \vee a_{\sigma(k)} \tag{3}$$

where the brackets stand for determinants and the sum is taken over all the permutations σ of $\{1, 2, \ldots, k\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(n-j)$ and $\sigma(n-j+1) < \sigma(n-j+2) < \cdots < \sigma(k)$. This formula, called the *shuffle formula*, is a very useful tool in practical applications.

It can be proved that if $A \neq 0$ and $B \neq 0$ and $\bar{A} \cup \bar{B}$ spans V, then

$$\overline{A \wedge B} = \bar{A} \cap \bar{B}. \tag{4}$$

An important property is that $\vee$ and $\wedge$ are dual operators. If we interchange $\vee$ and $\wedge$ we must interchange $V^{(k)}$ with $V^{*(n-k)}$.

We combine the vector spaces $V^{(k)}$ into another vector space over $\Re$,

$$\Lambda(V) = V^{(0)} \oplus V^{(1)} \oplus \cdots \oplus V^{(n)}$$

where $V^{(0)}$ and $V^{(n)}$ both coincide with $\Re$. The elements of $\Lambda(V)$ are all tensors, that is, arbitrary linear combinations of extensors of various steps. We have that $dim(\Lambda(V)) = \sum_{k=0}^{n} \binom{n}{k} = 2^n$.

The Grassmann-Cayley algebra is defined as the vector space $\Lambda(V)$ with the operations $\vee$ and $\wedge$. These operations are both associative, distributive over addition, and anticommutative.

3. Projective Representation of Twists and Wrenches

If $\mathbf{v}$ is the velocity of the Euclidean point $\mathbf{p}$, then we define the *motion* of the projective point p as $M(p) = (\mathbf{v}, \mathbf{p} \cdot \mathbf{v})^t$.

If a and b are finite projective points, for each point p in space we can express this motion in projective terms as $M(p) = (\mathbf{v}, \mathbf{p} \cdot \mathbf{v})^t = a \vee b \vee p$. The 2-extensor $a \vee b$, that represents the line passing through a and b, is called the *center of the motion*. Since $M(a) = 0$ and $M(b) = 0$, it represents a *rotation* about the axis determined by a and b. For example, the center of motion of a counterclockwise rotation having unitary angular velocity about the z axis can be represented by the extensor $\vee \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} = (0, 0, 1, 0, 0, 0)^t$, using the projective points $a = (0, 0, 1, 1)^t$ and $b = (0, 0, 2, 1)^t$.

A *translation* can be described as a rotation about an axis at infinity. Let $c = (c_1, c_2, c_3, 0)^t$ and $d = (d_1, d_2, d_3, 0)^t$ be two points at infinity, then the extensor $c \vee d$ can be used as the center of motion such that $M(p) = c \vee d \vee p$. The corresponding velocity is $\mathbf{v} = (c_2 d_3 - c_3 d_2, c_3 d_1 - c_1 d_3, c_1 d_2 - c_2 d_1)^t$. Since it is independent from the point p, it can be used to represent a translation. For example, a translation along the positive direction of the z axis having unitary linear velocity can be expressed by means of the two points at infinity $c = (1, 0, 0, 0)^t$ and $d = (0, 1, 0, 0)^t$. Then, the corresponding center of motion is the extensor $\vee \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = (0, 0, 0, 0, 0, 1)^t$.

With a serial composition of translations and rotations a more general screw motion in space can be obtained. This motion is represented by a *twist* $t = (\boldsymbol{\omega}, \mathbf{v})^t$, where $\boldsymbol{\omega}$ and $\mathbf{v}$ represent an angular velocity vector and a linear velocity vector, respectively. In general, a twist cannot be represented using the join of two projective points. However, it can be expressed as the composition of a translation along an axis and a rotation about the same axis. Instantaneously, the serial composition of twists corresponds to their simple addition.

A vector $\mathbf{f} = (f_1, f_2, f_3)^t$ that represents an Euclidean *force* applied at the Euclidean point $\mathbf{p} = (p_1, p_2, p_3)^t$ can be represented in the projective space by the join of the two projective points $p = (p_1, p_2, p_3, 1)^t$ and $f = (f_1, f_2, f_3, 0)^t$, that is, by the extensor $F = p \vee f = (\mathbf{f}, \mathbf{p} \times \mathbf{f})^t$. If we have a set of forces $F_i = p \vee f_i$ applied at a given point p, the resultant, $\mathbf{G}$, can be obtained by adding the free vectors f_i and applying this sum at p, that is, $G = \sum_i (p \vee f_i) = p \vee (\sum_i f_i)$.

If two forces $F_1 = p \vee f$ and $F_2 = q \vee g$ with $\mathbf{f} = -\mathbf{g}$ are applied to two distinct points p and q, the resultant $G = F_1 + F_2 = p \vee f + q \vee g = p \vee f + q \vee (-f) = (p - q) \vee f$ is called a *couple*. Since $p - q = (p_1 - q_1, p_2 - q_2, p_3 - q_3, 0)^t$ is a point at infinity, a couple can be thought as a force at infinity.

In general, the composition of forces in space does not correspond to a single new force and cannot be expressed using the extensor of two projective points. Their resultant is a *wrench* $w = (\mathbf{f}, \mathbf{m})^t$, where $\mathbf{f}$ and $\mathbf{m}$ are a force vector and a moment vector, respectively. A wrench can always be rewritten as a composition of a force along a line and a couple on the plane normal to the line.

We will represent twists using 2-extensors of $V^{(2)}$ while wrenches will be represented by means of the 2-extensors of its dual space $V^{*(2)}$.

Instantaneously, the resultant of the parallel composition of wrenches corresponds to their simple addition.

4. Twist-Wrench Duality

Let us consider a rigid body, say M, partially constrained by another, say S. We define the *twist space* $\mathbf{T}$ of M as the vector space of all possible instantaneous twists that it can have with reference to S, and we define the *wrench space* $\mathbf{W}$ as the vector space of all reaction forces that can be generated in the interaction between M and S.

Any twist $t \in \mathbf{T}$ of M must be *reciprocal* to any wrench $w \in \mathbf{W}$ between M and S because the power generated by t and w must be zero (see Duffy (1990)), that is, $\boldsymbol{\omega} \cdot \mathbf{m} + \mathbf{v} \cdot \mathbf{f} = \mathbf{0}$. This relation is nothing more than the expression of the pairing of the vector spaces $V^{(2)}$ and its dual $V^{*(2)}$

as expressed by (1). Thus, we can say that twist and wrench spaces of a partially constrained rigid body are dual spaces in the sense of Sect. 2.2.

Now, it should be clear that the kinestatic analysis of serial and parallel robot manipulators can be done in terms of 2-extensors, that is, in terms of the 6-dimensional vector spaces $V^{(2)}$ and $V^{*(2)}$.

If the twist space of a robot is the entire $V^{(2)}$, its end effector has full mobility while, if its wrench space is the whole $V^{*(2)}$, it can resist any wrench applied by the environment without exerting any force or torque at its joints. In general, since the twist space is a subspace of $V^{(2)}$ and the wrench space is a subspace of $V^{*(2)}$, it is convenient to set $V^{(2)} = H$ and work in the Grassmann-Cayley algebra $\Lambda(H)$ over this auxiliary vector space in which 2-extensors of $V^{(2)}$ become 6-dimensional vectors.

It is well known (see, for example, Bruyninckx and De Schutter (1993)) that the twist (wrench) space of the serial (parallel) combination of motion constraints is the sum of the twist (wrench) spaces of the composing constraints. Analogously, the twist (wrench) space of the parallel (serial) combination of motion constraints is the intersection of the twist (wrench) spaces of the composing constraints. It is important to point out that the reciprocity relation remains valid under serial and parallel combination of motion constraints.

These considerations can be re-formulated in the language of the Grassmann-Cayley algebra for kinematic chains, for which the centers of motion of their links are called *joint extensors*. From properties (2) and (4) of the join and meet operators it follows that:

- The twist (wrench) space of the serial (parallel) connection of kinematic chains is the support of the join of the extensors that represent the twist (wrench) spaces of the chains, provided that their twist (wrench) extensors are linearly independent.
- The twist (wrench) space of the parallel (serial) connection of kinematic chains is the support of the meet of the extensors that represent the twist (wrench) spaces of the chains, provided that the sum of the composing twist (wrench) spaces spans H (H^*).

For example, let $L_1, L_2, \ldots, L_k$ be linearly independent joint extensors of a serial manipulator R_{ser}. The twist space of R_{ser} is the support of the k-extensor $L_{R_{ser}} = L_1 \vee L_2 \vee \cdots \vee L_k$ of $H^{(k)}$, that is, $\mathbf{T}_{R_{ser}} = \overline{L_{R_{ser}}} = \overline{L_1 \vee L_2 \vee \cdots \vee L_k}$, where L_i are vectors of H. The wrench space is its dual, that is, the support of the $(6-k)$-extensor $L^*_{R_{ser}} = (L_1 \vee L_2 \vee \cdots \vee L_k)^* = L_1^* \wedge L_2^* \wedge \cdots \wedge L_k^*$ of $H^{*(6-k)}$, that is, $\mathbf{W}_{R_{ser}} = \overline{L^*_{R_{ser}}} = \overline{L_1^* \wedge L_2^* \wedge \cdots \wedge L_k^*}$. In this expression L_i^* are 5-extensors of $H^{*(5)}$.

Likewise, let $L_1, L_2, \ldots, L_k$ be the extensors of the kinematic chains that compose a parallel manipulator R_{par}. The wrench space of R_{par} is

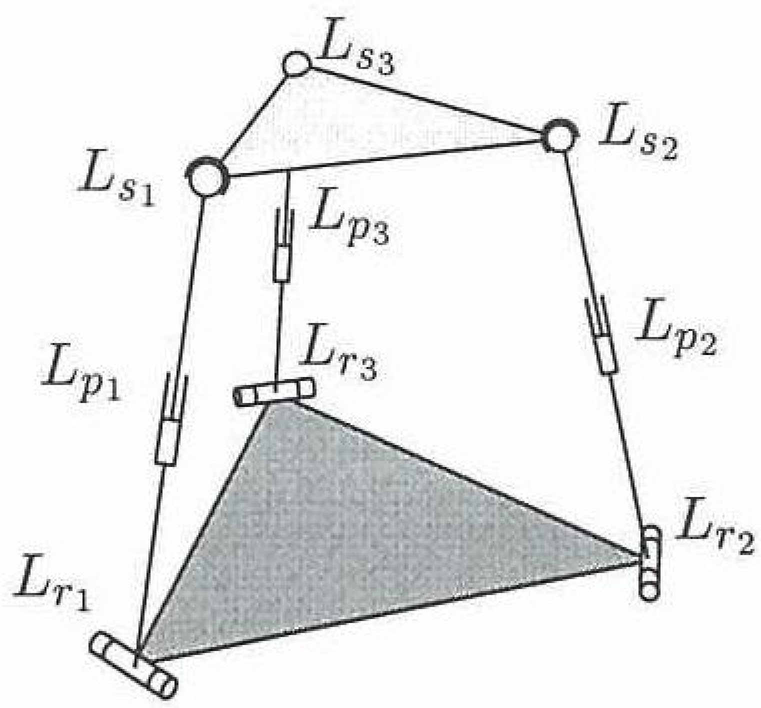

Figure 1. A parallel manipulator

the support of the extensor $L^*_{par} = L^*_1 \vee L^*_2 \vee \cdots \vee L^*_k$, that is, $\mathbf{W}_{R_{par}} = \overline{L^*_{R_{par}}} = \overline{L^*_1 \vee L^*_2 \vee \cdots \vee L^*_k}$. The corresponding twist space is the support of the extensor $L_{R_{par}} = L_1 \wedge L_2 \wedge \cdots \wedge L_k$, that is, $\mathbf{T}_{R_{par}} = \overline{L_{R_{par}}} = \overline{L_1 \wedge L_2 \wedge \cdots \wedge L_k}$.

5. An Example

In this section we describe a simple application of the Grassmann-Cayley algebra to the kinestatic analysis of a parallel manipulator, with special reference to the shuffle formula (3) for the meet operator.

Let us consider the parallel manipulator represented in Fig. 1 (Lee and Shah (1988)), where the spherical joints of the upper platform and the revolute joints of the lower one are both located at the vertices of an equilateral triangle.

The twist space $\mathbf{T}_i$ of the composing serial chain i is the support of $L_i = L_{r_i} \vee L_{p_i} \vee L_{s_{i,1}} \vee L_{s_{i,2}} \vee L_{s_{i,3}}$, where L_{r_i}, L_{p_i} are the rotational and the prismatic joint extensors, respectively. The spherical joint of the chain i is modelled as three revolute joints having intersecting axes whose extensors are $L_{s_{i,1}}$, $L_{s_{i,2}}$ and $L_{s_{i,3}}$. Its wrench space $\mathbf{W}_i$ is the support of L^*_i.

In a non-singular configuration, the twist space $\mathbf{T}$ of the parallel manipulator corresponds to the support of $L = L_1 \wedge L_2 \wedge L_3$. By using the associative property of the meet operator, the shuffle formula gives

$$\begin{aligned} L' = L_1 \wedge L_2 = &+ [L_{r_1} L_{r_2} L_{p_2} L_{s_{2,1}} L_{s_{2,2}} L_{s_{2,3}}] L_{p_1} \vee L_{s_{1,1}} \vee L_{s_{1,2}} \vee L_{s_{1,3}} \\ &- [L_{p_1} L_{r_2} L_{p_2} L_{s_{2,1}} L_{s_{2,2}} L_{s_{2,3}}] L_{r_1} \vee L_{s_{1,1}} \vee L_{s_{1,2}} \vee L_{s_{1,3}} \\ &+ [L_{s_{1,1}} L_{r_2} L_{p_2} L_{s_{2,1}} L_{s_{2,2}} L_{s_{2,3}}] L_{r_1} \vee L_{p_1} \vee L_{s_{1,2}} \vee L_{s_{1,3}} \\ &- [L_{s_{1,2}} L_{r_2} L_{p_2} L_{s_{2,1}} L_{s_{2,2}} L_{s_{2,3}}] L_{r_1} \vee L_{p_1} \vee L_{s_{1,1}} \vee L_{s_{1,3}} \\ &+ [L_{s_{1,3}} L_{r_2} L_{p_2} L_{s_{2,1}} L_{s_{2,2}} L_{s_{2,3}}] L_{r_1} \vee L_{p_1} \vee L_{s_{1,1}} \vee L_{s_{1,2}}. \end{aligned}$$

L' is a 4-extensor so that $L' \wedge L_3$ gives a 3-extensor. This means that, in this case, the support of L is a three dimensional subspace of H and hence the manipulator has three degrees of freedom. The wrench space $\mathbf{W}$ of this manipulator is the support of the 3-extensor L^*. In some cases, given the joint extensors, it is convenient to compute first $L^* = (L_1 \wedge L_2 \wedge L_3)^* = L_1^* \vee L_2^* \vee L_3^*$ whose support is $\mathbf{W}$ and then, by duality, $\mathbf{T}$.

6. Conclusions

In this paper a unifying framework for the study of the statics and the instantaneous kinematics of serial and parallel robot manipulators, based on the Grassmann-Cayley algebra, has been introduced. A projective interpretation of the concepts of twist, wrench, twist space and wrench space based on the concept of extensor has been presented. It has been shown that the join and meet operators are effective tools for joining and intersecting the linear subspaces involved in the kinestatic analysis of manipulators. Moreover, the duality inherent in the Grassmann-Cayley algebra has been used to reflect the duality between reciprocal twists and wrenches.

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