

# Motions of Bipartite Frameworks

by Walter Whiteley

At the special session on rigidity at Syracuse last Fall, the question was raised: must a spatial framework contain triangles to be rigid? This problem leads naturally to the problem of rigidity of frameworks with a bipartite graph  $K_{m,n}$ . The joints form two sets A and B (A with m elements, B with n elements) and the  $n \times m$  bars connect every joint in A with every joint in B (no bars between pairs in A or pairs in B). In the article Bolker and Roth present an elegant solution to the problem of rigidity for such bipartite frameworks in any dimension.

In space, frameworks with graphs  $K_{4,6}$ ,  $K_{5,5}$  or larger are in general statically rigid. However if all its joints lie on a quadric surface (a sphere, a hyperboloid, two planes, etc.) then a bipartite framework will not be rigid. In **Figure 1** we illustrate the case of a bipartite framework in the plane with joints on a circle. The illustrated radial velocities preserve infinitesimally the length of any bar joining A and B along a chord of the circle. Clearly this type of motion extends to a non-rigid motion for joints on a sphere in space. Since plane conics are all projectively equivalent, and infinitesimal motions are projectively invariant, the motion when all joints are on a circle carries over to the case when the joints are on any other plane conic. For spatial quadrics other than the sphere we can follow the motion in a plane cross section, and in that way find the velocities, which remain normal to the surface of the quadric.

A spatial model of the structure  $K_{4,6}$  having an infinitesimal motion can easily be built using only two lengths of bars, in the ratio of 1 to 1.93185. First

construct a tetrahedron using two of the shorter bars end-to-end along each edge. Connect each of these midpoints to the two opposite vertices of the tetrahedron, using the longer bars. This places all the joints on a sphere. We made a "popping" model of this  $K_{4,6}$  by cutting dowels in the ratio 1 to 2.05 and inserting them into rubber Unistrut nodes, thereby changing the effective ratio to 1 to 1.88. The popping model appears to be rigid under most test loads, and it is in practice quite difficult to find the permitted motion. But once the model is held correctly, it moves with ease.

In **Figure 2** we illustrate two cases of a bipartite framework with all joints on two lines. Once again the illustrated velocities preserve all bar lengths in the plane and it is easy to see this construction as the cross-section of a spatial construction for the joints on a degenerate conic of two planes or even a cone.

A second type of failure for a spatial bipartite framework occurs when one set A lies in a single plane and this plane either contains a joint from B or has all the joints of A on a conic of the plane. The reader may be able to develop appropriate motions for these cases as well.

The authors prove that these are the only possible types of failure using a detailed analysis of the number of static stresses in a realization of a bipartite graph in any dimension.

The analysis proceeds via an unusual linear transformation applied to stresses

$$T(\dots, w_{ij}, \dots) = (\dots, g_i, \dots)$$

$$\text{where } g_i = \sum w_{ij}$$

(sum over all j with edge (i,j) entering joint i). This transformation really separates the equilibrium at joint i:

$$\sum w_{ij} (a_i - a_j) = 0$$

into a linear dependence of the projective coordinates of the joints

$$\sum w_{ij} a_j = (\sum w_{ij}) a_i = g_i a_i.$$

In particular, for bipartite frameworks the kernel of  $T$  (all  $g_i = 0$ ) is generated simply as a tensor product of linear dependencies of the projective points in A with dependencies of the projective points in B:

$$\text{Ker } T = D(A) \times D(B).$$

For a bipartite framework the image of  $T$  is composed of linear dependencies among tensors  $axa$  of the joints. ( $axa$  is a vector with  $n(n+1)$  divided by 2 components, namely all products of pairs of projective coordinates

$$axa = (a_1 a_1, a_1 a_2, \dots, a_n a_{n+1}, a_{n+1} a_{n+1}).$$

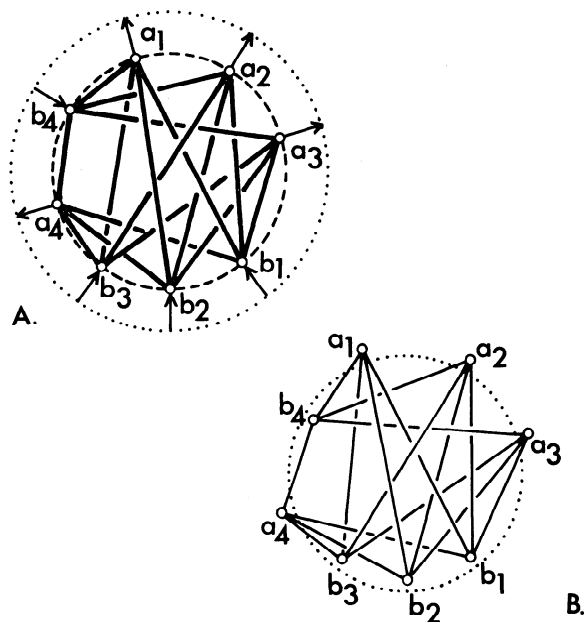
The dependencies making up the image of  $T$  are exactly those among tensors of joints in a special subset C, defined as follows. If we write the projective space spanned by A as A then  $C = (A \cup B) \cup (A \cup B)$ . The final step in their proof is to use a

matrix with these tensors as columns and then to study the column rank by counting row dependencies plus a standard count of the columns minus the rows. The scalars of the row dependencies are exactly the coefficients of the equations of quadric surfaces in  $\bar{C}$  which contain all points in  $C$ . These form a space  $Q(C)$ .

If  $C$  has  $K$  elements and  $\dim(\bar{C}) = h$ , then the central theorem is: for a bipartite framework in  $n$ -space the dimension of the space of stresses is

$$\dim(D(A))\dim(D(B)) + \dim(Q(C)) + k - (h + 1)(h + 2)/2$$

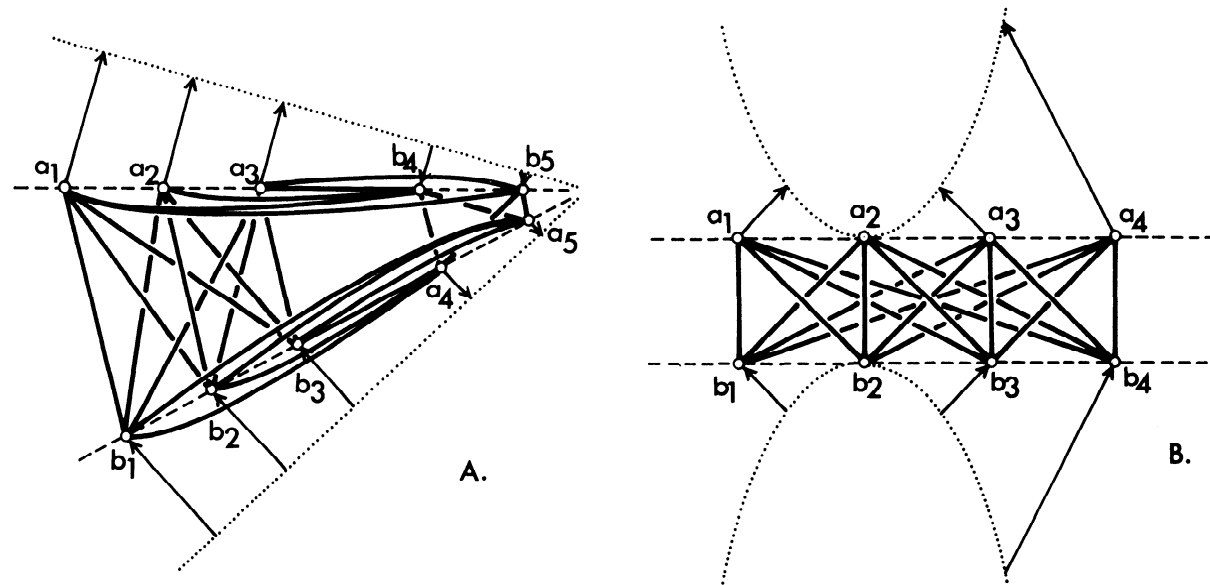
The results for the rigidity of spatial bipartite frameworks are obtained from this theorem by simple counting arguments.



**Figure 1.** If the joints of a bipartite framework lie on a circle (A) then the illustrated radial velocities give a permitted non-rigid infinitesimal motion of the framework (B).

These theorems raise several interesting problems. Given a realization of  $K_{4,6}$  in space, **is there a simple construction in projective geometry (forming planes from 3 points and points from 3 planes etc.) which checks whether the 10 joints lie on a quadric surface?** In the plane Pascal's theorem provides a check for 6 points on a conic, but we have not found a comparable result in space.

For a framework which counts to be isostatic (e.g. in 3-space  $E = 3V - 6$ ) the condition for non-rigidity can be written as a standard equation in the projective coordinates of the joints (a determinant of the rigidity matrix). It is possible to count the degree of this equation in each joint (in space, the valence less 2), but the authors' results yield an exact factoring of the condition for some simple frameworks. For  $K_{4,6}$  the



**Figure 2.** A bipartite framework with the joints on two lines will have a non-rigid infinitesimal motion. If the two sets are mixed together on the two lines, then the type of velocities illustrated in (A) give such a motion. If the two sets are separated with one line each, then the type of velocities in (B) give a non-rigid motion.

condition is factored as the expression for a general quadric times the square of the bracket  $(a_1 a_2 a_3 a_4)$ . We also know that if the framework forms a subgraph of a bipartite graph (if all cycles of bars and joints are even) then the framework will have a non-trivial motion when the joints lie on a quadric surface **What does this tell us about the form of the projective condition for such frameworks?**

Finally if a bipartite framework in space has all its joints in  $A$  on one plane and all the joints in  $B$  on a second plane, the central theorem guarantees 3 degrees of infinitesimal freedom. The type of construction illustrated in **Figure 2B** will give 2 of these degrees of freedom when extended along two planes through  $a_2 b_2$ . **Find a construction for velocities in the additional degree of freedom.**

Bolker 1979

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P—M—R

When is a bipartite graph a rigid framework?

Preprint, University of Massachusetts (Boston), Boston MA021-25, and University of Wyoming, Laramie, WY 82071.

Static stress and infinitesimal rigidity of bipartite frameworks. Projective conditions for dependence. Structures whose nodes lie on quadric surfaces.