Kinematic Analysis of Slider-Cranks Derived from the λ-Mechanism

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Abstract In this paper a kinematic analysis is presented for slider-cranks derived from the λ-mechanism. In particular, for this linkage the coupler curves traced by a reference point are Berard curves. By properly choosing the design parameters of the mechanism the coupler curves are represented by quartics, which have been identified and classified.

Key words: Kinematics, Slider-Crank, Coupler Curve, Singularity Analysis.

1 Introduction

Planar mechanisms are widely used in industrial environment for automatic machinery in order to give prescribed law of motions. They can be referred as *Function Generators* from the input to the output links, *Rigid Body Guidance* through the study of the rigid coupler motion and *Path Generators* by referring to the coupler curve, as extensively reported in [?] to [?]. This paper deals with the analysis of slider-cranks derived from the λ-mechanism, which can be used to give suitable coupler curves for application in automatic machinery, providing some constraints, such as the region in which the curve should be contained, or geometrical characteristics of the curve. Given a set of tasks constituting functional requirements, the design process consists in producing a mechanism that will meet all the specifications. Sometimes, for industrial applications, a first requirement deals with the definition of a suitable working area in which the mechanism should produce a given trajectory. This can be due to physical limits, or actuation constraints. Then a designer should select a shape of the path and other constraints, which can be linked to

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design parameters of the mechanism and constitute the design problem dealing with precision points, or given trajectory, or rigid body motion. Following this idea we propose design guidelines focusing on a slider-crank derived from the λ-mechanism [?]. In particular, it has been shown that this slider-crank has a practical engineering interest since it can better approximate a straight line than the corresponding 4-bar λ-mechanism [?].

2 Position Analysis of the Slider-Crank and Coupler Curves Expression

The first analytical investigation for a coupler curve of a four-bar linkage was undertaken by Prony [?], who analyzed Watt's straight-line motion (1796). Samuel Roberts showed in 1876 that the "three-bar curve" (coupler curve) of the four-bar linkage is an algebraic curve of the sixth order [?]. Cayley gave further properties of the curve. His interest was directed to linkages hypothetically able to generate specific algebraic curves of any order [?]. In general, the more links, the higher is the degree of curve generated. Since a curve can have up to as many intersections with a straight line as the degree of its polynomial expression, it is hypothetically possible to generate (or approximate) any trajectory designing a suitable mechanism. In this contest we focus our attention on slider-cranks that in general have fourth order coupler curves. The equation of the coupler point curve for any slider-crank mechanism may be obtained by analytic geometry being the loci of any point P that belongs to a segment for which a point B is constrained to lie on a circle and another point C is constrained to have a linear trajectory. In the following we restrain the attention on a particular slider-crank for which point P lies on the same line of the coupler link, as shown in the scheme of Fig. ??. It is known as a λ-mechanism [?]. The derivation presented here follows that of Samuel Roberts proposed for a 4-bar linkage [?]. The equation can be written in Cartesian coordinates, when the X axis is chosen along the line parallel to the slider, without loss of generality. Let (x, y) (s, e) be, respectively, the coordinates of coupler point P and point C, then

$$\begin{bmatrix} s \\ e \end{bmatrix} = \begin{bmatrix} x + (a_3 + w)cos(\theta_3) \\ y + (a_3 + w)sin(\theta_3) \end{bmatrix} \tag{1}$$

Since B describes a circle (or arc of a circle) centered in O with $\|OB\|^2 = a_2^2$

$$\|OB\|^2 = (x + wcos\theta_3)^2 + (y + wsin\theta_3)^2 \tag{2}$$

Let us take the second equation from ?? and ??. The coupler curve can be obtained by eliminating θ_3 from equations ?? and ??.

$$y - e + (w + a_3)sin\theta_3 = 0 \tag{3}$$

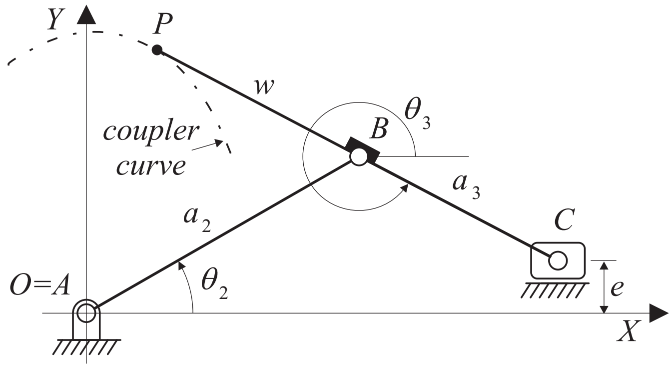

Fig. 1 A kinematic scheme of a slider-crank derived from λ-mechanism with design parameters.

$$x^2 + y^2 + w - a_2^2 + (2xw)cos\theta_3 + (2yw)sin\theta_3 = 0 \tag{4}$$

Let us consider the half-tangent substitution in **??** and **??** as

$$cos\frac{\theta_3}{2} = \frac{1-u^2}{1+u^2}, sin\frac{\theta_3}{2} = \frac{2u}{1+u^2}, \tag{5}$$

The coupler curve f of the slider-crank can be described by the algebraic equation of fourth-order in the form

$$\begin{aligned} f = (a_3-w)^2y^4 + (a_3+w)^2x^4 + 2(a_3^2+w^2)x^2y^2 + 4we(a_3-w)y^3 + \\ +4we(a_3-w)x^2y + [-2(a_3+w)^2(a_3^2+w^2) + 4w^2e^2]x^2 + \\ +[-2(a_3^2-w^2)(a_2^2-w^2) + 4w^2e^2]y^2 - 4we(a_2^2-w^2)(a_3+w) + \\ +(a_3+w)^2(a_2-w)^2(a_2+w)^2 \end{aligned} \tag{6}$$

f is symmetric with respect to the Y axis. This can be proved since Eq. **??** contains only even powers of x.

3 Characterization of the coupler curve

In general, a singularity is a point at which an equation, curve, or surface, becomes degenerate. Singularities are often called singular points or geometric singularities [**?**]. Real geometric singularities of the coupler curve of a slider-crank can be found considering f together with its partial derivatives f_x and f_y with respect to x, and y respectively. The zeros of the set of equations: $f = 0, f_x = 0$ and $f_y = 0$ gives the geometric singularities of the coupler curve. They can be identified as

$$x_s = 0;\ y_s = \frac{-ew \pm \sqrt{w^4 + (e^2 - a_3^2 - a_2^2)w^2 + a_2^2 a_3^2}}{a_3 - w} \tag{7}$$

Equation **??** can be used to identify geometric singularities of the coupler curve and can be further used to derive kinematic considerations. It is evident from eqs.**??** that singularities for this slider-crank may arise on the Y-axis only. Points belonging to these zeros are denoted by C_i, D_i, and A_i, when C_i indicates cusps, D_i double points and A_i acnodes. They can be classified by considering the second partial derivatives of f in the form **??**

$$g = f_{xy}^2 - f_{xx} - f_{yy} \tag{8}$$

Functions g, f_x and f_y, can be used to fully characterize real geometric singularities of the coupler curve. They can be related to a classical problem in linkages design known as a "branch defect". In particular, the presence of singularities gives information on the number of branches of the coupler curve, furthermore it allows the determination if two task positions lie on the same branch, as described in [**?**].

Equations **??** to **??** lead to a further investigation on the coupler curves characteristics by giving an enumeration of quartic equations representing the trajectory traced by the P reference point. In particular, we focus our attention on quartics studied by Berard in 1820 and then by Ruiz-Castizo in 1889. The coupler curve of a slider-crank derived from a λ-mechanism can be represented by Berard curves [**?**]. In particular they become "egg shaped" curves if the eccentricity is equal to zero. Furthermore, by properly choosing design parameters, these curves can be identified as well known quartics. In the following we investigate the properties of the coupler curves and their special cases.

-*Quartics of Bernoulli* (1687):
the curve traced by a coupler reference point P of a λ-slider-crank can be represented by Bernoulli quartics if the following conditions are met: $w = a_3$ and $e = 0$. The coupler curves in **??** then become

$$x^4 + x^2y^2 - 2(a_2^2 + a_3^2)x^2 + (a_2 + a_3)^2(a_2 - a_3)^2 = 0 \tag{9}$$

Quartics in **??** can be always represented by two affinely finite branches, as it can be proven they are free of geometric singularities. Examples of coupler curves representing Bernoulli quartics are shown in Fig.**??**a).Furthermore if, additionally, $a_3 = a_2$ Eq. **??** degenerates into a circle centered in the (0;0) with radius equal to $2a_2$, as it is shown in **??**a).

-*Quartics of Ruiz-Castizo*:
the curve traced by a coupler reference point P of a λ-slider-crank can be represented by Ruiz Castizo quartics if the following conditions are met: $a_3 = a_2 + e$ and $w = a_3$. The coupler curves in **??** can be expressed as **??**. Quartics in **??** can be always represented by one connected component, as the existence of geometric singularities can be proven.

$$x^4 + x^2y^2 - (4a_2^2 + a_2e + e^2)x^2 + e^2y^2 + (4a_2e^2 + 2e^3)y + \\ + e^2(4a_2^2 + a_2e + e^2) = 0 \quad (10)$$

Examples of coupler curves representing Ruiz-Castizo quartics are shown in Fig.**??**b). In particular, geometric singularities are given by

$$x_s = 0;\; y_s = -(2a_2 + e) \quad (11)$$

Singular point given in Eq.**??** and Fig. **??**b) is a cusp, since g in **??** is equal to zero. In addition, the curve in **??** degenerates if $e = -a_2$. If $e = 0$ Eq.**??** is a circle centered in the origin with radius equal to $2a_2$, as shown in Fig. **??**b). -*Lemniscate*:
the curve traced by a coupler reference point P of a λ-slider-crank can be represented by lemniscate if the following condition is met: $e \geq a_3 - a_2$. It can be always represented by one connected component, as the existence of geometric singularities can be proven. Examples of coupler curves represented by lemniscates are shown in Fig.**??**a). Geometric singularities are given by

$$x_s = 0;\; y_s = -(2a_2 + w);\; y_s = \frac{a_2w + a_2a_3 - w^2 - a_3w}{a_3 - w} \quad (12)$$

If $a_2 = a_3 = e = w$, then the curve in **??** then become

$$x^4 + y^2w^2 + x^2y^2 - 3w^2x^2 = 0 \quad (13)$$

-*Cardano motion*:
the curve traced by a coupler reference point P of a λ-slider-crank can be represented by Cardano motion if the following condition is met: $a_2 = a_3$ and $e = 0$.

$$[x^2 + y^2 - (a_2 + w)^2][(a_2 + w)^2x^2 + (a_2 - w)^2y^2 + (a_2^2 - w^2)^2] = 0 \quad (14)$$

In this case the coupler curves are obtained by the union of a circle and an ellipse, as shown in Fig.**??**b. Major semi-axis of the ellipse is on Y axis (X axis) if w is greater (less) than a_2. If additionally $w = a_2$ then the coupler curve represented by the ellipse degenerates into a circle. Geometric singularities of the curve are given by $y_s = \pm(a_3 + w), x_s = 0$. They are cusps.

4 Design Guidelines

The boundary curve of a family can be obtained by considering the equation of the family together with the derivative of the family with respect to the parameter. In particular, in the following coupler curves equation in **??**

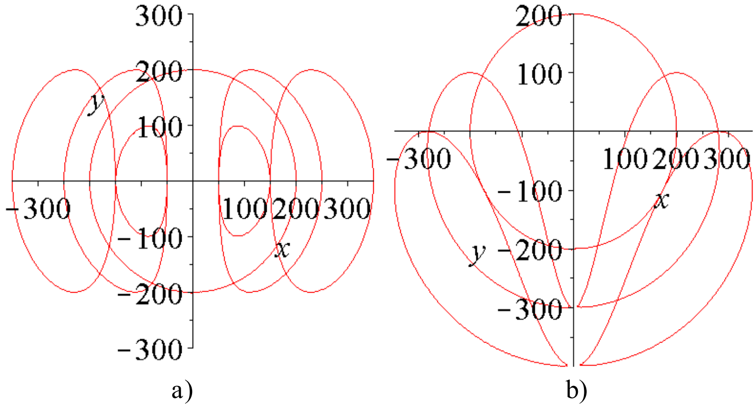

Fig. 2 Examples of Bernoulli quartics in a) and Ruiz-Castizo quartics in b).

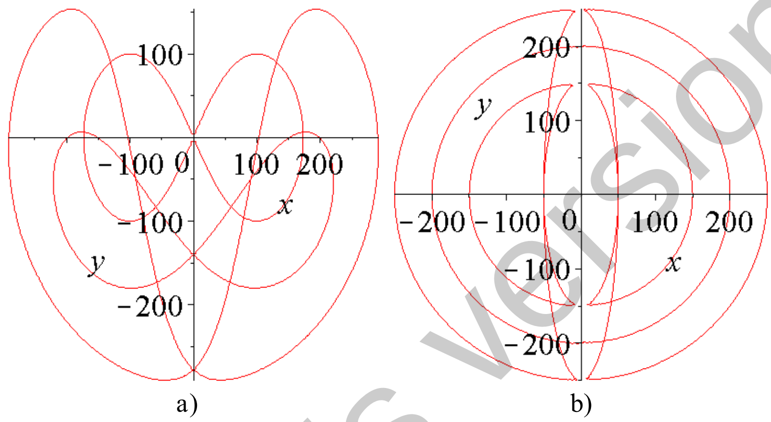

Fig. 3 Examples of lemniscates in a) and Cardano motion curves in b).

is considered, taking as a parameter of the family the eccentricity and coupler length, respectively, as shown in the numerical examples shown in Figs. **??** and **??**. If one considers eccentricity as the family parameter, then the equation of the boundary curve becomes, (when a_3 is different from $-w, w$ different from 0)

$$(w^2 + 2a_2 w + a_2^2 - x^2 - y^2)(w^2 - 2a_2 w + a_2^2 - x^2 - y^2) = 0 \tag{15}$$

If one considers a_3 as the family-parameter, then the equation of the boundary curve becomes, (when w is different from 0)

$$(w^2 + 2a_2 w + a_2^2 - x^2 - y^2)(w^2 - 2a_2 w + a_2^2 - x^2 - y^2) = 0 \tag{16}$$

According to the above-mentioned considerations, design guidelines of a slider-crank can be given in terms of shape and characteristics of the coupler curves as follows:

1. define limits for y and x in the plane of motion, say x_{max}, x_{min}, y_{max}, y_{min};
2. evaluate design parameters in Eqs **??** and **??**;
3. the family parameter can be chosen according to other design specifications.

In particular, in this context we can generate any symmetrical egg shaped path giving overall size of the curve in the plane of motion OXY, being characterized for example to have a coupler point with stationary curvature,

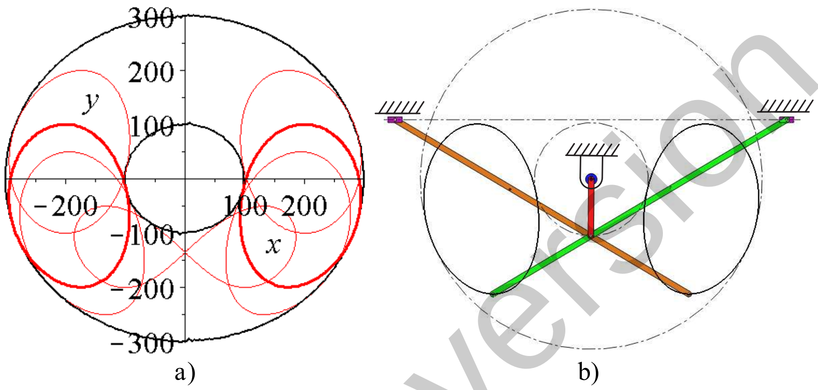

Fig. 4 Boundary curve and family curve of e parameter when $a_2 = 100$, $a_3 = 400$, $w = 200$ in a) and an example of designed slider-crank with $e = -100$.

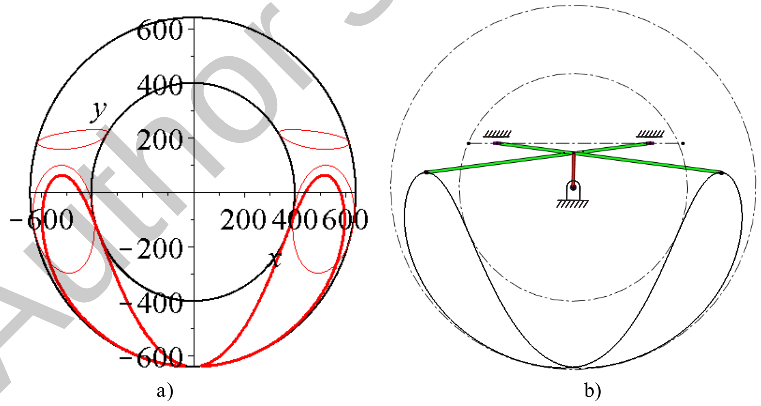

Fig. 5 Boundary curve and family curve of a_3 parameter when $a_2 = 120$, $e = 150$, $w = 270$ in a) and an example of a designed slider-crank with $a_3 = 520$.

when a suitable reference point P is chosen along the coupler link BC, as belonging to the cubic of stationary curvature $\mathcal{C}$. In fact, for the crank angle $\theta_2 = 0$, the cubic of stationary curvature degenerates in a ϕ-curve, as reported in [?]. Moreover, when P is chosen as coincident with the inflection pole J, or in general on the inflection circle, an approximate straight line can be also obtained, as reported in [?].

5 Conclusion

By studyng the kinematics of slider-cranks derived from λ-mechanism giving the coupler curve described by a fourth-order polynomial, this paper provides interesting characteristics processed by the classical slider-crank linkage when parameters of the mechanism are properly chosen. In particular, the coupler curve traced by a reference point of the coupler can be represented by quartics of Bernoulli, quartics of Ruiz-Castizo, lemniscate and producing Cardano motion. Furthermore, in this paper singularities of the coupler curve are investigated.

References

1. McCarthy J.M., Joskowicz L., 2001, Kinematic synthesis, Cambridge University Press.
2. Hall A.S., 1961, Kinematics and Linkage Design, Waveland Press. Inc., Prospect Heights, Illinois.
3. Hunt K.H., 1990, Kinematic Geometry of Mechanisms, Oxford University Press., New York.
4. Shoup T.E., 1991, Centrodes of the Slider-Crank Mechanism, 8th IFToMM World Congress on the Theory of Machines and Mechanisms, Prague, Vol.1, pp. 59-62.
5. Dijksmann, E.A.,1995, The Inverted Slider-Crank used for the design of an approximate Straight-Line Mechanism, Engineering Research Bd. 61, pp.129-134.
6. Prony, G. F., 1796, Nouvelle architecture hydraulique, voI. II, Paris.
7. Norton R.L., 1999, Design of machinery, McGraw-Hill.
8. Hartenberg R.S. and Denavit J., 1964, Kinematic synthesis of linkages McGraw Hill.
9. Gibson C.G., 1998, Elementary Geometry of Algebraic Curves, Cambridge Univ. Press.
10. Schrocker H.-P., Husty M. L., McCarthy J. M., 2005, Kinematic Mapping Based Evaluation of Assembly Modes for Planar Four-Bar Synthesis, Proc. of ASME 29th Mechanism and Robotics Conference, Long Beach, DETC2005/MECH-85037.
11. Figliolini G., Conte M., Rea P., 2012, Algebraic algorithm for the kinematic analysis of crank/rocker mechanisms, ASME J. of Mechanisms and Robotics, vol. 4, no. (1) 011003-1.
12. Dijksmann, E.A., 1976, Motion Geometry of Mechanisms, Cambridge University Press., London.