Determination of Maximal Singularity-Free Workspace of Parallel Mechanisms Using Constructive Geometric Approach

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Abstract This paper proposes a novel approach to obtain the maximal singularity-free regions of planar parallel mechanisms which is based on a constructive geometric reasoning. The proposed approach consists of two algorithms. First, the borders of the singularity-free region corresponding to an arbitrary start point of the moving platform is obtained. Then, the second algorithm aims to find the center of the maximal singularity-free circle which is obtained using the so-called offset curve algorithm. As a case study, the procedure is applied to a 3-PRR planar parallel mechanism and results are given in order to graphically illustrate the effectiveness of the proposed algorithm. The proposed approach can be directly applied to obtain the maximal singularity-free circle of similar parallel mechanisms, which is not the case for other approaches proposed in the literature which is limited to a given parallel mechanism, namely, 3-RPR. Moreover, as the main feature of the proposed approach, it can be implemented both in a CAD system or in a computer algebra system where non-convex and re-entrant curves can be considered.

Key words: Parallel mechanisms, Singularity-free workspace, Geometric approach, Offset curve algorithm.

1 Introduction

Parallel Mechanisms (PMs) are a type of robotic mechanical systems composed of one moving platform and one base connected by at least two serial kinematic chains in parallel [8]. PMs are often erroneously said to be recent developments, have a pedigree far more ancient than that of serial manipulators, which are usually called

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anthropomorphic [3]. However, the last two decades have witnessed a noticeable rise in the number of publications regarding the kinematic and dynamic analyses of PMs to the end of proposing the most promising design. As it will be seen later on, PMs have their own drawbacks and even a simple one can lead to complicated kinematics analysis. In general, when a PM tends towards structural generality, its geometry and kinematic analysis get more complicated. The latter is the case of this paper where upon applying a simple modification into the kinematic arrangement of a planar 3-Degree-Of-Freedom (DOF) planar PM, the so-called 3-R$\underline{\text{P}}$R PM, and turning it to a 3-$\underline{\text{P}}$RR PM, then the problem of singularity-free workspace becomes a cumbersome task and would be elusive to classical approaches, proposed for the former 3-R$\underline{\text{P}}$R PM [6, 7, 10]. Here and throughout this paper, $\underline{\text{P}}$ stands for an actuated prismatic joint and R stands for a passive revolute joint.

Designing a PM with a singularity-free workspace is a vital condition for further analysis, such as path planning and control problem. In the literature, most of the study propounded on this topic, i.e., singularity-free workspace, are based on either primitive numerical approach or some complicated mathematical approaches where both entail some limits. In [2], Bonev *et al.* conducted an exhaustive study on the singularity locus of planar 3-DOF PMs by resorting to *screw theory*. In [10], a method based on geometric parameters of the mechanism under study is represented for which a singularity-free circle in the workspace of a 3-R$\underline{\text{P}}$R PM is obtained. In [7], Li *et al.* redefined the problem as an optimization problem accompanied with a constraint and resorted to Lagrangian multipliers and obtained the maximal singularity-free circle of a 3-R$\underline{\text{P}}$R PM for a prescribed center point. In [6], Jiang and Gosselin proposed some numerical recipes in order to find the singularity-free workspace of planar 3-DOF PMs.

This paper aims at obtaining the Maximal Singularity-Free Circle (MSFC) of 3-DOF planar PMs for a given orientation of the mobile platform. Obtaining the MSFC has eminent effect on reliability and endurance of the workspace of the robot. The circle is chosen because it has the most regular shape and comes in handy in practice. To the best knowledge of authors, in the literature, results of the MSFC were obtained only for a prescribed center point and this assumption bounds the radius of the circle and results into a local optimum solution. In this study, the center point of the MSFC is not prescribed from the outset and subject to be found using the geometrical reasoning proposed in this paper. It should be noted that the MSFC can be readily computed once the center is obtained. The proposed approach for obtaining the center point of the MSFC is based on a novel constructive geometric procedure which is the unique aspect of this work and distinct it with the others reported in the literature [7, 10].

In this paper, a novel geometric algorithm is proposed, called Alg. I, in order to obtain the singularity-free region of PMs which could be applied to non-convex singularity locus. Moreover, offset curve algorithm, Alg. II, is adapted for the geometric purpose of this work. Offset curve algorithm [1, 4] is a geometric constructive tool which has diverse engineering applications and has consequently motivated extensive researches concerning various offset techniques. It plays an important role in numerical controls and CAD/CAM applications [4]. To the best knowledge of the authors, the problem of MSFC has never been investigated upon a geometric standpoint. The proposed algorithm, which is inspired from geometric properties associated to the MSFC, could be implemented either in a computer algebra system or using a CAD system. Almost all the CAD systems have the possibility to make an offset of complex curves. in this paper, due to the simplicity, the details are skipped. Thus, more emphasis is placed on the numerical approach proposed in this paper to make such an offset to a given curve, specifically the singularity-locus curve.

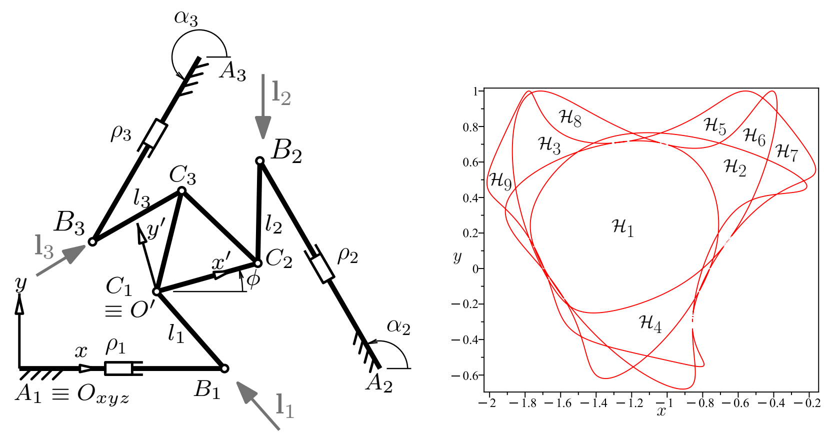
Through this paper, in order to illustrate the proposed approach, as a case study, the procedure of obtaining the MSFC is applied to a 3-PRR planar PM. However, it can be extended to all planar 3-DOF PMs presented in [2]. To the best knowledge of authors, 3-RPR and 6-UPS (SPS) PMs have been widely treated in the literature since they lead respectively to quadratic and cubic polynomial expressions for their singularity locus which simplifies considerably the mathematical challenge. A minor modification in the kinematic arrangement, for instance having a 3-PRR PM, leads to the complexity of the procedure for which methods reported in [7] are no more applicable and fail to provide satisfactory results. One of the problems in such an investigation is the presence of the square roots in the singularity loci expressions. The proposed algorithm is split into two sub-algorithms: (1) a first algorithm to obtain the subregion of interest for the MSFC, called Alg. I, and (2) a second one for obtaining the center point of the MSFC for the foregoing subregion, called Alg. II.

The remainder of this paper is organized as follows. First, the kinematic properties of the PM under study, i.e., the 3-PRR PM, is broadly reviewed. Then, Alg. I toward obtaining the singularity-free region is fully described, by having in mind that, as a case study, it will be applied to the 3-PRR PM. Finally, the offset curve algorithm is introduced to the sake of proposing Alg. II, which is applied into the singularity region obtained from Alg. I.

2 Kinematic Review of 3-PRR Planar Parallel Mechanism

A 3-PRR planar PM consists of three kinematically identical limbs actuated by a prismatic joint fixed at the base and followed by two passive R joints, as depicted in Fig. 1(a). As it can be observed from Fig. 1(a), O_{xyz}, with $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ as unit vectors, represents the fixed frame and O_{xyz} stands for the moving frame. The pose (position and orientation) of the mechanism is defined by (x, y, ϕ) where $\mathbf{p} = [x, y]^T$ and ϕ represent respectively the Cartesian position and the orientation of the moving frame with respect to the fixed frame. Upon resorting to *screw theory* [2], the Jacobian matrix of the mechanism can be formulated as follows:

$$\mathbf{J} = \left[\begin{array}{ccc|ccc} \mathbf{l}_1 & \mathbf{l}_2 & \mathbf{l}_3 & \mathbf{0} & \mathbf{0} & \hat{\mathbf{k}} \\ \mathbf{r}_1 \times \mathbf{l}_1 & \mathbf{r}_2 \times \mathbf{l}_2 & \mathbf{r}_3 \times \mathbf{l}_3 & \hat{\mathbf{i}} & \hat{\mathbf{j}} & \mathbf{0} \end{array}\right]^T, \tag{1}$$



(a) 3-PRR planar PM [2]

(b) Singularity locus of 3-PRR planar PM for $\phi = \pi/36$ and subregions $\mathscr{H}_i$

Fig. 1 Representation of (a) the schematic and (b) the singularity locus of a 3-PRR planar PM.

in which $\mathbf{l}_i$, $i = 1,2,3$, is the unit vector along the line connecting point B_i to point C_i and $\mathbf{r}_i$ is the vector connecting the origin of the moving platform to point C_i. Singular configurations of the mechanism occurs when the Jacobian matrix becomes rank deficient [5], i.e., the determinant of the foregoing matrix vanishes, $\det(\mathbf{J}) = 0$. The latter leads to have a polynomial of degree 20 (20 in y and 16 in x) for a constant-orientation of the moving platform [2]. It is worth to be noticed that the latter polynomial corresponds to all the eight working modes of the mechanism and, as reported in [2], it is not possible to find a polynomial expression for a single working mode among the eight one. It should be noted that obtaining such a polynomial is an extremely delicate task and is beyond the scope of this paper. Skipping the latter mathematical manipulations, Fig. 1(b) depicts the singularity locus of the 3-PRR planar PM for $\phi = \pi/36$.

3 Algorithm to Obtain the Subregion of the Singularity-free Workspace, Alg. I

As it can be inferred directly from Fig. 1(b), the singularity locus is such that splits the workspace of the mechanism into different regions which, in this paper, are referred to as *subregion* and called $\mathscr{H}_i$, $i = 1, \cdots, n$. It should be noted that some subregions are not mentioned in Fig. 1(b) to not overload the figure. This section is devoted to present a new method to the end of obtaining the boundaries of the singularity-free subregion, $\mathscr{H}_i$. It is worth noting that the proposed method could be applied to any kind of complex curve and it does not depend on the convexity of

Algorithm 1 The pseudo-code of the algorithm to obtain the subregion of the singularity-free workspace, Alg. I.

Input: $\det(\mathbf{J}) = 0$, P_0 as the starting point of the moving platform and ε as the desired accuracy
Output: The corresponding singularity-free subregion, called $\mathscr{H}_1$, consists of P_i, $i = 1, \cdots, n$
$i \leftarrow 1$;
$P_i = \texttt{fminsearch}(|\det(\mathbf{J})|, P_0)$ % use "Nelder-Mead" to find a point on the singularity locus
while $|P_i - P_{i-1}| < \varepsilon$ **do**
    $i \leftarrow i + 1$
    $K_i = \texttt{circle}(P_i, \varepsilon)$ % create a clockwise circle with P_i and ε as center and radius
    $S = \texttt{solve}(\det(\mathbf{J}), K_1)$ % save the intersection points of the circle and the singularity locus
    $P_i = \texttt{order}(S, clockwise)(1)$ % save the first item of S with respect to trigonometry order
end while

the subregions. Moreover, the main challenge in finding a subregion is the intersection points among different branches of the singularity curve, which are known as *bifurcation* points, called B as indicated in Fig. 2.

Algorithm 1 represents the concept of Alg. I which the reasoning is fully described in what follows. The first step to obtain the boundaries of a subregion is to specify which subregion among $\mathscr{H}_i$, $i = 1, \cdots, n$, is of concern. This can be done by specifying an arbitrary point, P_0, lying inside the desired subregion. In practice, this point is the position of the moving platform in the reference configuration of the mechanism. Therefore, the workspace of the moving platform should be bounded within the subregion of the start point, i.e., P_0.

The algorithm starts by finding a point on the singularity locus which lies on the boundary of the desired subregion, called point P_1. The latter can be done readily by using an unconstrained optimization approach for $|\det(\mathbf{J})| = 0$, as the objective function, i.e., using direct pattern search, namely Nelder-Mead (simplex) method with ε as simplex parameter [9]. From P_1, the algorithm starts to search through the boundary of the subregion other points constituting the subregion. To do so, a line is passed from P_0 to P_1 and creates a trigonometry circle, K_1, with P_1 and ε as center point and radius, respectively. The value of ε stands for the computation accuracy. The trigonometric circle covers angle between $\phi = [0, 2\pi)$ and can be either clockwise or counter-clockwise. The line corresponds to $\phi = 0$. By changing the angle t from 0 to 2π, the first intersection point of K_1 and singularity locus will be saved and called P_2, as depicted in Fig. 2. In practice, this can be done by considering the discrete circle. By the same token, a trigonometric circle, called K_2, will be created with P_2 and ε as center point and radius, respectively, with the same direction as the previous circle. The same procedure pursues for new points P_i, $i = 1, \dots n$, and at each step the first intersection point will be added to a list of points, called $\mathscr{C}_0$. The stopping criterion of the algorithm is that, the last obtained point, P_n, be close enough to the first member of $\mathscr{C}_0$, i.e., P_1. In other words, $\|P_n - P_1\| < \varepsilon$. Finally, $\mathscr{C}_0$ is a closed polygon which represents the singularity-free region corresponding to the reference configuration of the mechanism.

The main feature of this algorithm is its ability to deal properly with the multi-sectional areas caused by intersection among the singularity curve. This type of

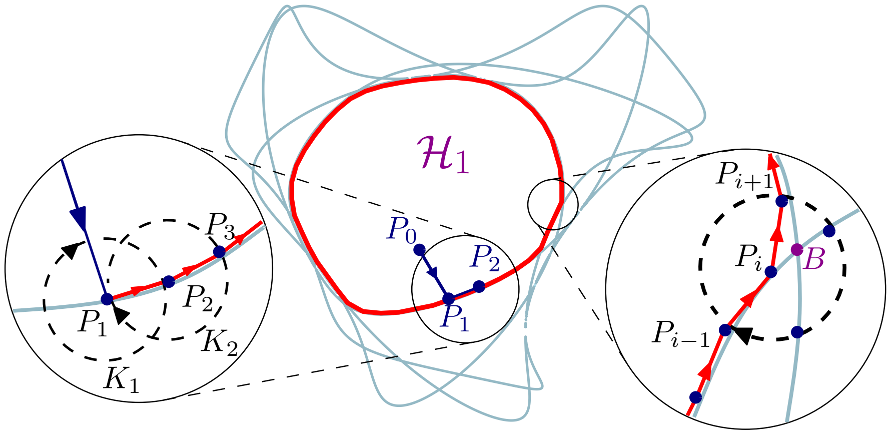


Fig. 2 Result of applying the proposed algorithm to the end of obtaining the singularity-free region of $\mathscr{H}_1$. P_0 is the starting point and the the red polygon represents the singularity-free of $\mathscr{H}_1$.

areas are represented in Fig. 2, and, as it can be observed, due to the reasoning of the algorithm, these multi-sectional areas have no significant effect on the procedure and will be automatically circumvented. More precisely, Alg. I is able to detect the correct region when approaching a bifurcation point, B. In Fig. 2, Alg. I is applied in order to find $\mathscr{C}_0$ as the singularity-free region. Point P_0 is the position of the moving platform in its reference configuration. By resorting to Nelder-Mead (simplex) method, $\varepsilon = 0.5$, a point close to the singularity locus is obtained, P_1. Pursuing Alg. I, more points P_i, $i = 1, \cdots, n$, are obtained. **It took 3 sec to compute $\mathscr{C}_0$, with a 2 GHz processor**. In fact, $\mathscr{C}_0$ will be used in the next section as the singularity-free region in order to obtain the MSFC. As it will be more apparent in the upcoming section, errors due to the iterative approximation of singularity-free region $\mathscr{C}_0$ tend to zero upon applying offset curve algorithm.

4 Obtaining the MSFC Using Offset Curve Algorithm, Alg. II

The whole concept of the offset curve algorithm, is based on two geometric properties of MSFC for which it should be (a) tangent to the intersection points between the MSFC and the boundaries of the polygon $\mathscr{C}_0$ and (b) its center point should be equidistant to all the intersection points. For a closed-planar polygon $\mathscr{C}_j(t)$, its offset polygons can be written mathematically as follows [4]:

$$\mathscr{C}_{j+1}(t) \longleftarrow \mathscr{C}_j(t) \pm d\,\mathbf{n}(t), \quad j = 1,\, 2,\, \ldots m \tag{2}$$

where d is the offset distance and $\mathbf{n}(t)$ is the normal vector at point t on the polygon $\mathscr{C}_j(t)$. In the problem addressed in this paper, "$-$" is considered as $\pm$, because it is desired to decrease the area of $\mathscr{C}_i$ to a point. Having in mind the two latter properties

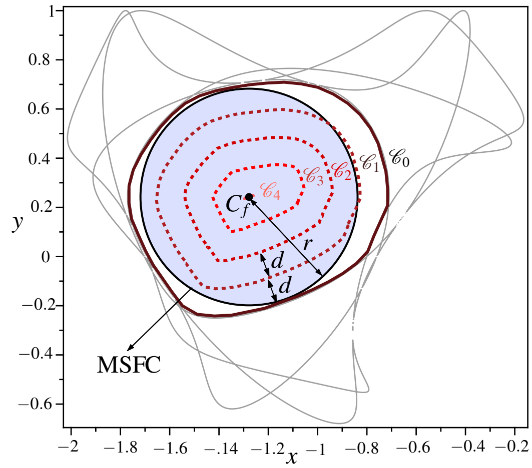


Fig. 3 Using Alg. I, singularity-free region $\mathscr{H}_1$ is obtained, called $\mathscr{C}_0$. Then, by applying four times Alg. II (offset concept) into $\mathscr{C}_0$, the area of the last polygon, $\mathscr{C}_4$, is less than ε. Therefore, it can be estimated by a point, C_f as the center of MSFC with radius as $r = 4\,d$.

of the MSFC, the algorithm is organized as follows. The first step consists in obtaining the tangent of each point t on the perimeter of the polygon $\mathscr{C}_0$. Then each point on the perimeter is moved forward by a given value, d, in the direction of the line perpendicular to its tangent which yields a new polygon, $\mathscr{C}_1$. By the same token, one can obtain $\mathscr{C}_2$, $\mathscr{C}_3$, ... and $\mathscr{C}_m$. The latter procedure should be persuaded in such a way that for a given m, the area of $\mathscr{C}_m$ reduces to approximately a point for which the algorithm stops. The latter point, C_f, represents the center of the MSFC. The radius of the circle is simply computed as $r = m\,d$, where m stands for the number of applied offset and can be chosen arbitrarily.

It should be noted that a special situations may arise, which consists in the cases for which the curve contains some necks. In such cases, upon pursuing the offset curve algorithm the curve will be separated and split into different curves and the algorithm should apply the offset approach for each subregion and obtain the corresponding MSFC [1, 4]. The MSFC is the biggest one for all the subregions.

The offset curve algorithm is available in Matlab by using the command `bufferm` and almost all CAD software have the capability of executing such a curve offset. Figure 3 represents the result of the MSFC for a 3-$\underline{P}$RR PM for a constant-orientation of the moving platform singularity locus. Using Alg. I, the singularity-free subregion of the mechanism for a prescribed orientation is obtained, $\mathscr{C}_0$. Then by applying Alg. II, the corresponding MSFC is obtained. In Fig. 3, $\mathscr{C}_i$, $i = 1, \cdots, 4$, are new offset polygons in which, each of them is generated by offsetting its preceding one by d as the normal distance. The latter is continued until converging to a point, C_f, being the center of the MSFC. Finally, by having the center of MSFC the corresponding radius can be readily obtained.

5 Conclusion

This paper proposed a new geometric constructive approach to the end of obtaining the singularity-free region and the maximal singularity-free circle of 3-DOF planar parallel mechanisms. The procedure consisted of two algorithms, which are mainly based on geometrical reasoning of the problem. First, using a new method the boundaries of the singularity-free region corresponding to the starting point of the moving platform was obtained. Then using the so-called offset curve algorithm, the center of the maximal singularity-free circle in the corresponding region was computed. Special conditions were taken into account and proved the robustness of the algorithm. As a case study, the 3-PRR planar parallel mechanism was considered. Ongoing works consist in extending the algorithm to higher DOF PMs and taking into account the workspace boundaries as additional constraints to the problem, which is a definite asset in practice.

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