

# A blend of Delassus four-bar linkages

Chung-Ching Lee and Jacques M. Hervé

**Abstract** In 1922, Delassus found out three four-bar linkages with four parallel helical H pairs whose one-degree-of-freedom mobility is conditioned by equalities of bar lengths. Our paper establishes that these linkages are not independent. Two isosceles triangle HHHPs are hybridized to obtain the Delassus 4H rhomboid (kite) linkage. Using an auxiliary chain whose mobility is explained by a group of Schoenflies motions and a Delassus 4H parallelogram, the Delassus 4H crossed parallelogram is newly derived from this rhomboid. It is further verified that the isosceles triangle and the Delassus parallelogram are two basic linkages and that the rhomboid and the crossed parallelogram stem from them. Finally, a blend of Delassus 4-bar linkages is proposed and can be used as a basic building block (BBK) for deployable structures. Two examples of deployable linkages with four and six BBKs are introduced.

**Key words:** Delassus linkage, isosceles triangle, rhomboid, parallelogram, Schoenflies motion, deployable linkage.

## 1 Introduction

Seeking after all the four-bar linkages implementing lower pairs, Delassus [1] discriminated two main categories of mobility, namely ordinary (ordinaire in French) and singular mobility. In four-bar linkages, the Degree of Freedom (DoF) may be not modified if a link is cut in two parts which are welded after a relative motion of the parts. When any cut-and-weld operation does not alter the mobility, all the relative motions are subsets of only one subgroup of the rigid-body displacement group

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and the linkages are "trivial" (or banal) chains in the classification by displacement subgroups [2]. In exceptional chains, the effect of a cut-and-weld operation depends on the choice of the link. In "singular" linkages of Delassus, the mobility is affected by a cut-and-welding of any one of its four links. The singular chains of Delassus are qualified as paradoxical. The paradoxical mobility is subject to geometric conditions that require the use of the Euclidean metric [3]. In trivial and exceptional chains, only concepts of 3D affine geometry are sufficient to stipulate the geometric constraints so the Euclidean metric is not necessarily employed.

Page 304 in Delassus' paper [1] summarizes the findings that there exist only five singular four-bar chains with one-DoF finite mobility. They are

1. The rhomboid (deltoid or kite) with parallel screw pairs at its four vertices; the pitches of the screws located on the diagonal axis are equal to each other and are equal to the arithmetic mean of the remaining two. If  $h_i (i = 1, 2, 3, 4)$  denote the pitches, we have:  $h_1 = h_3 = (h_2 + h_4)/2$ .
2. The parallelogram with four parallel screws; the sums of pitches of diagonally opposite screws are equal:  $h_1 + h_3 = h_2 + h_4$ .
3. The crossed parallelogram with four parallel screws; the pitches of alternate screws are equal:  $h_1 = h_3$  and  $h_2 = h_4$ .
4. Two parallel screws with equal pitches and two rectilinear sliding pairs bilaterally symmetric with respect to the plane containing the screw axes.
5. The Bennett linkage.

The first four Delassus linkages implement helical H pairs (or screws) with parallel axes. Hereinafter, they are called Delassus four-bar linkages. The relative motions between the links are 1-DoF motions included in the 4-D subgroup of Schoenflies motions whose axis direction is parallel to the screw axes. The 1-DoF mobility is subject to geometric conditions that imply scalar equation tying the link lengths and the pitches of screw pairs. The last one is the only movable four-revolute chain with non-parallel and non-intersecting revolute (R) joint axes. It was first discovered by G. T. Bennett in 1903 [4] and E. Borel [5] also found it independently only one year later. This linkage and the fourth one are beyond the scope of our article and we focus just on the first three chains.

It is worth mentioning that recently a new paradoxical four-bar linkage was disclosed in [6]. It is an isosceles triangle of structural type HHHP. The H pairs have parallel axes; the two HH links have equal lengths and the prismatic P pair is parallel to the triangle side with a variable length. The three H pitches satisfy the equality  $h_1 + h_3 = 2h_2$ . In this HHHP chain, a relative motion is a helical Cardan motion as explained in [7].

We notice that the 1-DoF mobility in the 4H rhomboid and the 4H crossed parallelogram is conditioned by two equalities of H pitches whereas the 4H parallelogram and the HHHP isosceles triangle have to satisfy only one pitch equality. In what follows, we will show that the two 4H linkages with two pitch equalities can be derived from the isosceles triangle and the parallelogram with one pitch equality.

Firstly, we hybridize two isosceles triangle HHHP linkages to obtain the rhomboid (kite) 4H linkage. Secondly, in the rhomboid 4H loop, we add two links to form a second loop, which is a special Delassus parallelogram. The third loop ap-

pearing in the combination is the general crossed parallelogram of Delassus. In a similar way, we can construct a second crossed parallelogram. Finally, we obtain a blend of four-bars linkages with paradoxical 1-DoF mobility. Actually, the blend of two rhomboids produces a basic building block, which can be used as a module to construct complex deployable structures.

## 2 Isosceles triangle HHP linkage and its helical Cardan motion

In an isosceles triangle HHP linkage, the three pitches are tied by one scalar equality, namely  $h_1 + h_3 = 2h_2$ . Hence, each of the three pitches is a function of two independent numbers. It is convenient to adopt the numbers  $p$  and  $q$  to verify  $h_1 = q + p$ ,  $h_2 = q + p/2$  and  $h_3 = q \Leftrightarrow h_1 = q + p$ ,  $h_1 - h_3 = p$  and  $2h_2 = 2q + p$ . Using the numbers  $p$  and  $q$  provides a concise classification of the types of isosceles triangles. When  $p = 0$ , the three H pitches are equal and the 1-DoF mobility of the HHP linkage is not conditioned by the equality of the two HH link lengths; in other words, the triangle is not necessarily isosceles. Therefore, the paradoxical mobility which is assumed in the paper implies  $p \neq 0$  and  $q$  can have any value.

When  $p = q = 0$ , the H pairs are revolute R pairs and the corresponding isosceles triangle RRRP linkage is planar. As it is well known, if one body of the P pair is assumed to be fixed, then one RR link rotates around a fixed axis and the motion of the other RR link is Cardanic. When  $p \neq 0, \forall q$ , the isosceles triangle generates a noteworthy kind of motion named helical Cardan motion, which is presented for the first time in [6,7]. This new helical Cardan motion can be regarded as a spatial generalization of the Cardanic movement on a plane, shown in Fig. 1.

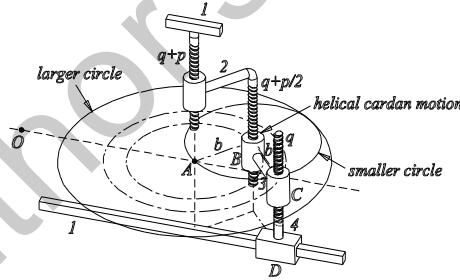


Fig.1 Isosceles triangle HHP linkage and 1-DoF helical Cardan motion

### 3 Formation of Delassus rhomboid 4H linkage

From two HHHP chains of the helical Cardan motion, we geometrically derive the rhomboid (deltoid or kite) 4H Delassus chain. The two isosceles triangles are not generally congruent but the homologous H pairs have equal pitches. Using self-explanatory figures, we can identify two distinct HHHP chains in Fig.2. The two chains have the same screw pitch  $q + p$  in the H pair of links 1 and 2 and the same screw pitch  $q$  in the H pair of links 3 and 4. The pitch of the screw pair between links 2 and 3 is assumed to be  $q + p/2$ . In one chain, the bars 2 and 3 have the same length  $a$ ; in the other chain, each length of bars 2 and 3 is  $c$ . These two distinct HHHP linkages can be merged in such a way that the resulting mechanism is the rhombus HHHH linkage. In the first step of the synthesis, two HHHP chains share a common P pair, as shown in Fig.3. Moreover, the two axes of the Hs with the pitch  $q + p$  coincide and the other two H pairs with the pitch  $q$  are also coaxial. In

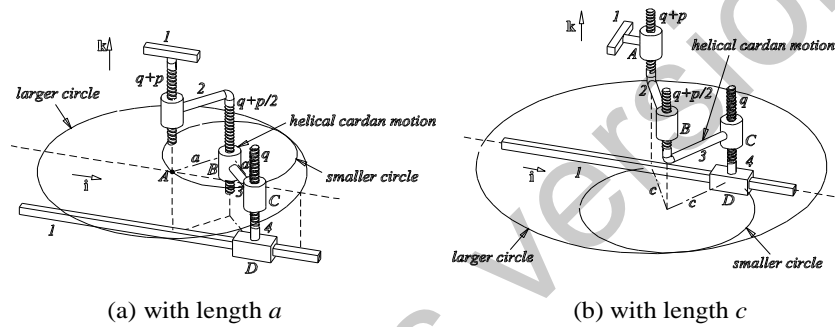


Fig.2 Two HHHP chains with the same pitches

Fig.3, actually, two coaxial Hs with the same pitch are equivalent to one H once we ignore the internal helical self-motion of the intermediate body. Consequently, the links 2 and 2 as well as the links 3 and 3 can be connected by only one helical pair with the same pitch. Furthermore, removing the P pair leads to a rhomboid HHHH chain, as shown in Fig.4. When  $p$  is equal to zero, all pitches are equal to  $q$  and the HHHH chain is movable whatsoever bar lengths are; the mobility of the chain can be derived using only the closure of the product in the pseudoplanar motion group [8]. When  $p$  is not zero, the movability is subject to metric conditions and is qualified as paradoxical. The special case with  $q = 0$  and  $p \neq 0$  gives an HHR paradoxical chain.

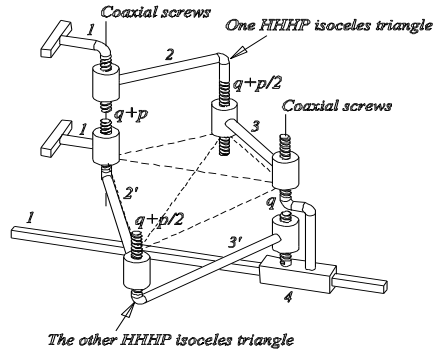


Fig.3 Two HHHP chains

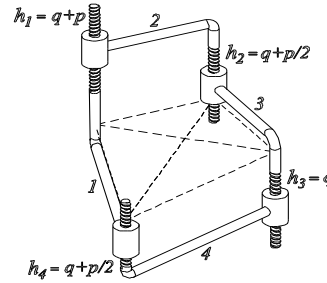


Fig.4 Rhomboid HHHH chain

#### 4 New derivation of crossed parallelogram and a blend of Delassus 4-bar linkages

In a Delassus 4H rhomboid, we add two bars, which are jointed by a cylindrical C pair whose axis is parallel to H axes. Each of the two H pairs with the same pitch  $q + p/2$  in the rhomboid forms a double H of pitch  $q + p/2$  with each of the two added bars. With the added bars, as shown in Fig.5, the resulting chain includes three closed loops. One loop is the original rhomboid and the other two have an HHC structural type. Any HHC chain with four parallel axes is a 1-DoF trivial chain for the Schoenflies motion group [9] whose axis direction is parallel to those of H axes. Consequently, the whole chain is movable with one DoF. Moreover, we assume also that, in one of the HHC subchains, the two CH bars are parallel to two adjacent bars of the rhomboid. That way, an HHC subchain forms a parallelogram. Then, the C pair can be replaced by an H pair with the pitch of the coinciding Delassus 4H parallelogram [10]. That pitch is computed by using the pitch equality of a Delassus parallelogram. In fact, we have two choices for constructing an HHC parallelogram. In one choice, the H pitches of the HHC parallelogram are  $q + p/2$ ,  $q + p$  and  $q + p/2$  and the C pair in Fig. 5(a) can be replaced by an H with the pitch  $q$ . A 4H crossed parallelogram appears as being a subchain of the whole chain and consequently moves also with one DoF. In two opposite Hs, the H pitches are equal, which is expressed by two independent equalities, namely  $h_2 = h_4 \Leftrightarrow q = q$  and  $h_1 = h_3 \Leftrightarrow q + p/2 = q + p/2$  with the notations of Fig.5(b). In the other choice, as shown in Fig. 6(a), the H pitches of the HHC parallelogram are  $q + p/2$ ,  $q$  and  $q + p/2$  and the C pair can be replaced by an H with the pitch  $q + p$ . A crossed parallelogram appears as being a subchain which moves also with one DoF. In two opposite Hs, the H pitches are equal, which is expressed by two independent equalities, namely  $h_1 = h_3 \Leftrightarrow q + p = q + p$  and  $h_2 = h_4 \Leftrightarrow q + p/2 = q + p/2$ , Fig.6(b). The crossed parallelogram of Delassus is derived from two 1-DoF chains, the Delassus 4H kite and the Delassus 4H parallelogram. Hence, it is a combination

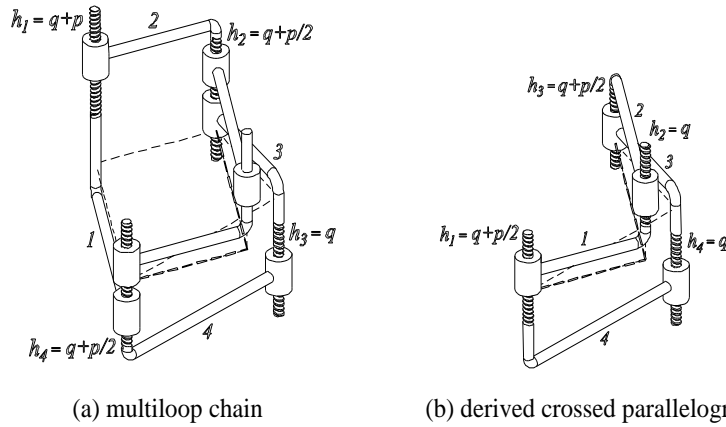


Fig.5 Forming HHHC subchain

of two HHHP isosceles triangles together with a Delassus parallelogram. In other words, the HHHP isosceles triangle and the Delassus 4H parallelogram imply the existence of the 4H kite and the 4H crossed parallelogram. Furthermore, we have

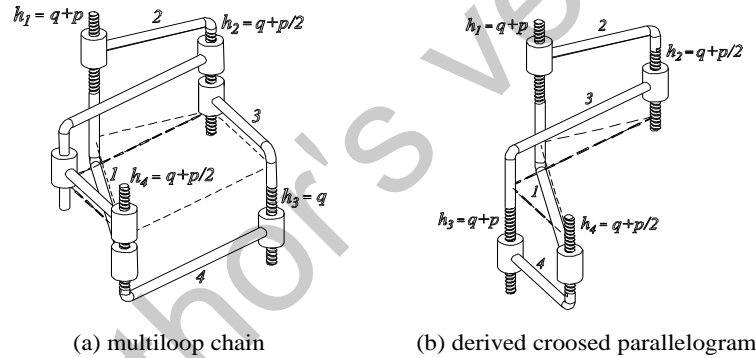


Fig.6 Second crossed parallelograms

synthesized a blend of four-bar linkages, in which each loop has a paradoxical 1-DoF mobility as shown in Fig.7 (isosceles triangles HHHP loops are not depicted). This multiloop linkage can be regarded as a basic building-block (BBK) to construct complex deployable structures. Fig.8 shows this kind of deployment by utilizing four BBKs and its distinct postures are displayed in Fig.9. Fig.10 illustrates the deployable ring structure composed of six BBKs.

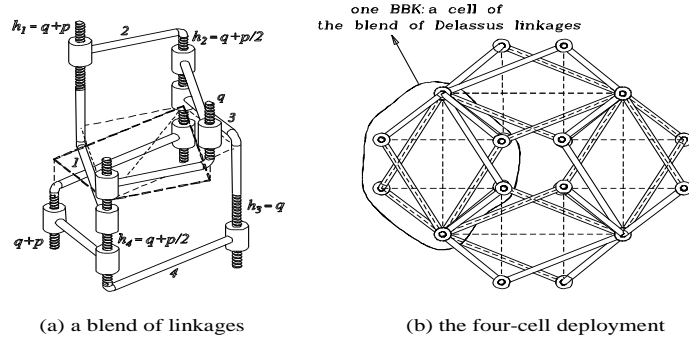


Fig.7 A blend of Delassus 4-bar linkages and its deployable structure

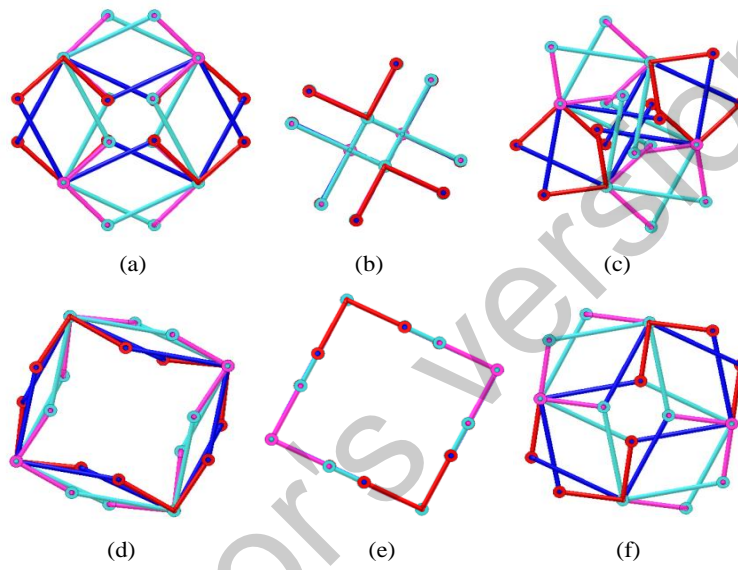


Fig.8 Different configurations of the four-cell deployment

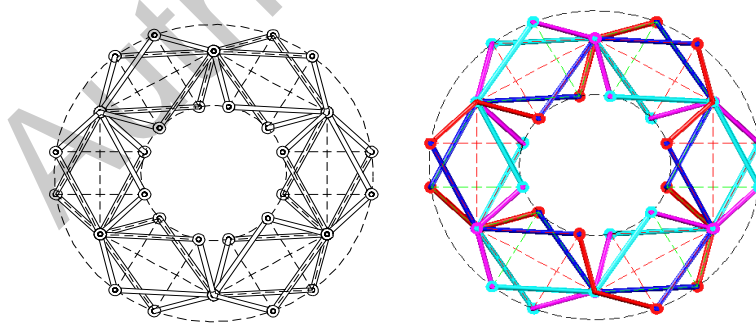


Fig.9 A deployable ring structure

## 5 Conclusions

The HHHP isosceles triangle and the Delassus 4H parallelogram are basic linkages. Both linkages imply the existence of the 4H kite and the 4H crossed parallelogram. It also shows that two Delassus four-bar linkages and the HHHP triangle are closely related. A blend of Delassus 4-bar linkages can further serve as a module in the construction of different-shaped deployable structures with screw joints. In these deployable structures, the pitches of all screw pairs can be reduced to being zeros. In practical application, deployment devices with revolute pairs are easier to manufacture than the ones with screw pairs but the former are suffering from the singularity when three revolute axes are coplanar. The latter can avoid the singular poses by the choice of their pitches. Hence, deployable devices with screw pairs may be more attractive for potential application.

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