# Non-existence of planar projective Stewart Gough platforms with elliptic self-motions

Dedicated to Prof. H. Stachel on the occasion of his 70th birthday by G. Nawratil

**Abstract** In this paper, we close the study on the self-motional behavior of non-architecturally singular parallel manipulators of Stewart Gough (SG) type, where the planar platform and the planar base are related by a projectivity $\kappa$, by showing that planar projective SG platforms with elliptic self-motions do not exist. The proof of this result demonstrates the power of geometric and computational interaction, but it also points out the limits of symbolic computation.

**Key words:** Self-motion, Stewart Gough platform, Borel Bricard problem



## 1 Introduction

The geometry of a planar Stewart Gough (SG) platform is given by the six base anchor points $\mathsf{M}_i$ located in the fixed plane $\pi_{\mathsf{M}}$ and by the six platform anchor points $\mathsf{m}_i$ of the moving plane $\pi_{\mathsf{m}}$. If the geometry of the manipulator and the six leg lengths are given, the SG platform is in general rigid, but under particular conditions, it can perform an $n$-parametric motion ($n > 0$), which is called self-motion. Note that these motions are also solutions of the famous Borel Bricard problem (cf. [1, 2, 3]).

It is well known, that planar SG platforms, which are singular in every possible configuration, possess self-motions in each pose (over $\mathbb{C}$). These so-called architecturally singular planar SG platforms were extensively studied in [4, 5, 6, 7]. Therefore, we are only interested in self-motions of planar SG platforms, which are not architecturally singular. Moreover, within this paper, we focus on the case, where the base anchor points $\mathsf{M}_i$ and the platform anchor points $\mathsf{m}_i$ are related by a non-singular projectivity $\kappa$. For the remainder of this article, we call these manipulators planar projective SG platforms.

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## 2 Self-motions of planar projective SG platforms

It is well known (cf. [6, 8, 9]), that a planar projective SG platform is architecturally singular, if and only if, one set of anchor points is located on a conic section, which can also be reducible.

The author proved in Lemma 1 of [10] that one can attach a two-parametric set $\mathscr{L}$ of additional legs to a planar projective SG platform without changing the forward kinematics and singularity surface. The platform anchor points $\mathsf{m}_i$ and the base anchor points $\mathsf{M}_i$ of these additional legs are also related by $\kappa$, i.e. $\kappa$: $\mathsf{m}_i \mapsto \mathsf{M}_i$.

Moreover, it was also shown by the author in [10] that non-architecturally singular planar projective SG platforms can either have pure translational self-motions or elliptic self-motions. Under consideration that $\mathsf{s}$ denotes the line of intersection of $\pi_{\mathsf{M}}$ and $\pi_{\mathsf{m}}$ in the projective extension of the Euclidean 3-space, the latter type of self-motions can be defined as follows (cf. Definition 1 of [10]):

**Definition 1.** A self-motion of a non-architecturally singular planar projective SG platform is called *elliptic*, if in each pose of this motion $\mathsf{s}$ exists with $\mathsf{s} = \mathsf{s}\kappa$ and the projectivity from $\mathsf{s}$ onto itself is elliptic.

Note, that an elliptic projectivity of a projectively extended line (line plus its ideal point) onto itself, is a bijective linear mapping, which does not have real fixed points. Therefore, Definition 1 implies that neither $\pi_{\mathsf{M}}$ and $\pi_{\mathsf{m}}$ nor two related points of the platform and the base coincide during an elliptic self-motion.

As the geometry of all manipulators with translational self-motions were already determined in [10], we focused on the study of elliptic self-motions in a recent publication [11], where the following results were obtained.

### 2.1 Results on elliptic self-motions

Until now, it is an open question, whether planar projective SG platforms with elliptic self-motions even exist (cf. later given Conjecture 1). In the case of existence, these self-motions have to be one-parametric ones with instantaneously two degrees of freedom in each pose of the self-motion (cf. Theorems 1 and 2 of [11]).

It was also shown in [11], that the angle $\gamma$ enclosed by the unique pair of ideal points $(\mathsf{f}, \mathsf{F})$ with $\mathsf{f}\kappa = \mathsf{F}$ has to remain constant during the self-motion of a planar projective SG platform. By introducing the nomenclature *orthogonal* for elliptic self-motions with $\gamma = \pi/2$, we can give Theorem 3 of [11]:

**Theorem 1.** *There do not exist non-architecturally singular planar projective SG platforms with an orthogonal elliptic self-motion.*

The proof of this theorem was done analytically, but not in the classical way (cf. Section 5.1 of [11]), as this approach resulted in a highly non-linear system of 17 equations in the design parameters, which we were not able to solve explicitly.

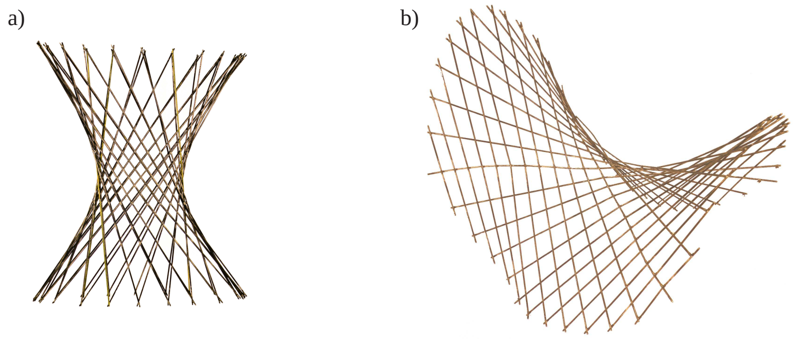


**Fig. 1** Wiener's models of a deformable one-sheeted hyperboloid (a) and hyperbolic paraboloid (b) of the collection of mathematical models at the Institute of Discrete Mathematics and Geometry, Vienna University of Technology (see http://www.geometrie.tuwien.ac.at/modelle).

Instead, we developed an alternative method (cf. Section 5.2 of [11]), which is based on the algebraic formulation of two geometrically necessary conditions for achieving an elliptic self-motion. These two conditions imply two homogeneous polynomials $\Upsilon_1$ and $\Upsilon_2$ of degree 12 in two Euler parameters $e_1$ and $e_2$, which remain from the Study parameters, after a performed elimination process. Note, that each of these two polynomials has 1960 terms. A necessary condition for the existence of an orthogonal elliptic self-motion is, that $\Upsilon_1$ and $\Upsilon_2$ are fulfilled independently of the Euler parameters $e_1$ and $e_2$. Therefore, the coefficients of $\Upsilon_1$ and $\Upsilon_2$, with respect to $e_1$ and $e_2$, imply a system of 26 equations in the design parameters, which was used to prove Theorem 1.

Due to the above cited results, we had good reasons to close the paper [11] with the following conjecture:

*Conjecture 1.* Non-architecturally singular planar projective SG platforms with an elliptic self-motion do not exist.

Clearly, the first idea to prove this conjecture, is to do it similarly to Theorem 1. Indeed, the problem under consideration has only one more unknown, namely the angle $\gamma$, but exactly this additional variable effects enormously the computational complexity: The two corresponding polynomials $\Upsilon_1$ and $\Upsilon_2$ of the alternative method can be computed with MAPLE on a high capacity computer (78GB RAM). Each of these two expressions has 8259 terms and is again of degree 12 in $e_1$ and $e_2$ (cf. Remark 4 of [11]). We tried hard to solve the resulting system of 26 equations explicitly, but we failed due to its high degree of non-linearity.

Therefore, we have to come up with another idea for proving this conjecture. This new approach, presented in Section 4, is a purely geometric one, which is based on some old geometric/kinematic results listed in Section 3. Note, that the given proof also finishes the study of planar projective SG platforms with self-motions, whose results are summed up within the conclusions (cf. Section 5).

## 3 Related historical work

Before we list related historical results, we repeat some elementary facts on regular ruled quadrics, which are the one-sheeted hyperboloid and the hyperbolic paraboloid (see Fig. 1): Both surfaces carry two sets of generators, which are called regulus $\mathscr{R}$ and associated regulus $\mathscr{R}^{\times}$, respectively. Moreover, it should be noted that all lines within one set are skew to each other and that each line of one set is intersected by all lines of the other set. Therefore, a regular ruled quadric is uniquely determined by three pairwise skew generators. For more details, we refer to [12].

In 1873 the following theorem was given by Henrici (cf. [13]):

**Theorem 2.** *If the generators of a hyperboloid $\Phi$ of one sheet are constructed of rods, jointed at the points of crossing in a way that at each intersection point one rod is free movable about the other one, then the surface is not rigid, but permits a deformation into a one-parametric set $\mathscr{H}$ of hyperboloids.*

Moreover, Greenhill remarked in 1878 that $\mathscr{H}$ consists of confocal hyperboloids and that the trajectory of a point of $\Phi$ is orthogonal to this system of confocal hyperboloids. A proof of Greenhill's statement was given in 1879 by Cayley [14]. In 1899 Schur [15] presented a very elegant proof for Henrici's theorem and Greenhill's addendum, which also showed that these results remain valid if the one-sheeted hyperboloid is replaced by a hyperbolic paraboloid.

Finally, it should be noted that Wiener, who made some very nice models of these deformable one-sheeted hyperboloids and hyperbolic paraboloids (see Fig. 1), also gave a detailed review of this topic in Section 9 of [16].

Beside the cited results on the deformation of regular ruled quadrics, the following theorem is well known to the kinematic community (cf. page 222 of [17]):

**Theorem 3.** *If three points $m_1, m_2, m_3$ of a line $g$ run on spheres, where the centers $M_1, M_2, M_3$ are also located on a line $G$, then every point $m$ of $g$ has a spherical trajectory, where the center $M$ of this sphere belongs to $G$ and fulfills the relation: $CR(m_1, m_2, m_3, m) = CR(M_1, M_2, M_3, M)$, where $CR$ denotes the cross-ratio.*

Moreover, it is a well known fact of projective geometry, that the one-parametric set of lines $[m, M]$ with $m$ and $M$ of Theorem 3 span a regulus $\mathscr{R}$ of a regular ruled quadric, if $g$ and $G$ are skew and $m_1, m_2, m_3$ and $M_1, M_2, M_3$ are pairwise distinct.

## 4 Proof of Conjecture 1

The proof of this conjecture is done by contradiction. We assume that a non-architecturally singular planar projective SG platform with base anchor points $M_1, \ldots, M_6$ and platform anchor points $m_1, \ldots, m_6$ exists, which possesses an elliptic self-motion $\mathscr{E}$. Without loss of generality, we can assume that the manipulator is in a pose of $\mathscr{E}$ where $\pi_m$ and $\pi_M$ are not parallel, as this would imply that $\kappa$ is an affinity. But this affine case was already discussed in Theorem 5 of [10].

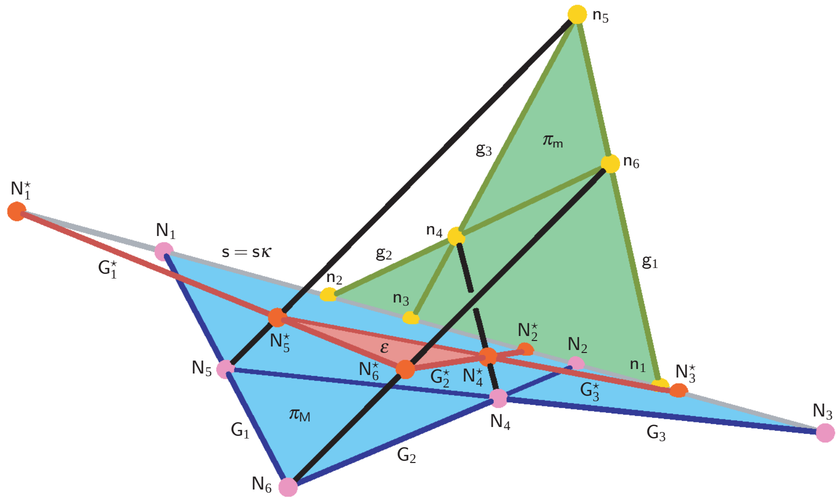


**Fig. 2** Sketch and notation of the points, lines and planes used for the proof of Conjecture 1.

### *4.1 Definition of a special planar projective SG platform*

Due to Lemma 1 of [10] and the results of [18], we can replace the original six legs $\overline{m_i M_i}$ with $i = 1, \ldots, 6$ by a new set of six legs $\overline{n_i N_i}$ without changing the direct kinematics and singularity surface, if $n_i \kappa = N_i$ holds and $n_1, \ldots, n_6$ are not located on a conic section. Therefore, $n_1, \ldots, n_6$ can be selected as follows (cf. Fig. 2):

We chose three lines $g_1, g_2, g_3 \in \pi_m$ in a way that $g_1, g_2, g_3, s$ are pairwise distinct and that no three of them belong to a pencil of lines. Then we can define $n_i$ as the intersection point of $g_i$ and $s = s\kappa$ for $i = 1, 2, 3$. Moreover, the intersection point of $g_i$ and $g_j$ is noted by $n_{k+3}$ with pairwise distinct $i, j, k \in \{1, 2, 3\}$. By applying $\kappa$ to $n_1, \ldots, n_6$, we get the corresponding base anchor points $N_1, \ldots, N_6$. It can easily be checked, that the resulting special planar projective SG platform is not architecturally singular. Moreover, we denote $g_i \kappa$ by $G_i$ for $i = 1, 2, 3$.

Now we consider the one-parametric set of legs $\overline{nN}$ with $n \in g_1$, $N \in G_1$ and $n\kappa = N$. Due to Lemma 1 of [10], all these legs $\overline{nN}$ can be added to the manipulator without disturbing the elliptic self-motion $\mathscr{E}$.[1] Moreover, the two lines $g_1$ and $G_1$ are skew ($\Leftrightarrow n_1 \neq N_1$), as the projectivity of $s$ onto itself is elliptic. As a consequence, the one-parametric set $\mathscr{R}_1$ of lines $[n, N]$ is a regulus of a regular ruled quadric $\Phi_1$. Due to the results of Henrici and Schur, we can even add arbitrary lines of the associated regulus $\mathscr{R}_1^\times$ to the mechanism without restricting the elliptic self-motion $\mathscr{E}$. Note, that the lines $g_1$ and $G_1$ also belong to $\mathscr{R}_1^\times$.

Clearly, analogous considerations for $g_i, G_i$ yield the corresponding results for the reguli $\mathscr{R}_i, \mathscr{R}_i^\times$ of the regular ruled quadric $\Phi_i$ for $i = 2, 3$.

[1] Note, that this can also be concluded as follows: As the cross ratio is invariant under projectivities the relation $CR(n_1, n_5, n_6, n) = CR(N_1, N_5, N_6, N)$ holds. Then Theorem 3 yields the results.

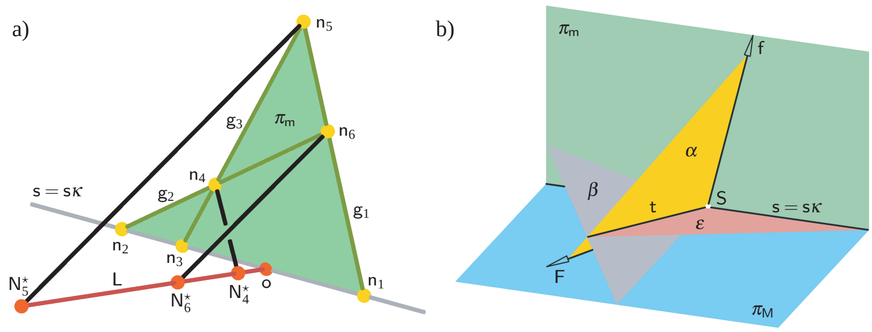


**Fig. 3** (a) Sketch for the proof of the second part of Lemma 1. (b) Sketch of the construction for the special choice of the plane $\varepsilon$.

## *4.2 Planar intersection of $\Phi_1$*

In the general case the planar intersection of $\Phi_1$ is a conic section. But, if we assume that the plane $\varepsilon$ contains the line $s = s\kappa$, which belongs to the regulus $\mathscr{R}_1$, then the conic degenerates into two distinct lines, which are $s = s\kappa$ itself and a line $G_1^\star$ which belongs to the associated regulus $\mathscr{R}_1^\times$. Note, that $\varepsilon$ is the tangent plane of $\Phi_1$ in the intersection point of $s = s\kappa$ and $G_1^\star$, which is denoted by $N_1^\star$.

Clearly, analogous considerations hold for the surfaces $\Phi_2$ and $\Phi_3$, which also yield the points $N_2^\star$ and $N_3^\star$, respectively. Moreover, we introduce the notation $N_{k+3}^\star$ for the intersection point of $G_i^\star$ and $G_j^\star$ with pairwise distinct $i, j, k \in \{1,2,3\}$. For these three points $N_4^\star, N_5^\star, N_6^\star$ the following statement holds:

**Lemma 1.** *The points $N_4^\star, N_5^\star, N_6^\star$ are pairwise distinct and do not belong to $s = s\kappa$. Moreover, $N_4^\star, N_5^\star, N_6^\star$ are not collinear.*

*Proof.* The point $N_{k+3}^\star$ is located on the line $[n_{k+3}, N_{k+3}]$, which belongs to the reguli $\mathscr{R}_i$ and $\mathscr{R}_j$ for pairwise distinct $i, j, k \in \{1,2,3\}$. Therefore, these three lines $[n_4, N_4]$, $[n_5, N_5]$, $[n_6, N_6]$ are pairwise skew and not located within the platform $\pi_m$. As a consequence, the points $N_4^\star, N_5^\star, N_6^\star$ are pairwise distinct and not located on $s = s\kappa$.

Now, we prove the second part of this lemma by contradiction. We assume that $N_4^\star, N_5^\star, N_6^\star$ are located on a line L (cf. Fig. 3a). We denote the intersection point of L and $s = s\kappa$ by o. It should be noted, that $o = N_1^\star = N_2^\star = N_3^\star$ holds. Moreover, L belongs to the associated regulus $\mathscr{R}_4^\times$ of the regular ruled quadric $\Phi_4$ defined by the regulus $\mathscr{R}_4$, which is spanned by the pairwise skew lines $[n_4, N_4]$, $[n_5, N_5]$ and $[n_6, N_6]$. Now, the unique line of $\mathscr{R}_4$ through o has to be $s = s\kappa$, as otherwise this point has to be a fixed point of the projectivity of s onto itself, which contradicts the definition of an elliptic self-motion. Therefore, the intersection of $\Phi_4$ with $\pi_m$ has to consist of $s = s\kappa$ and a second line containing the points $n_4, n_5, n_6$, which already contradicts our assumptions of Section 4.1. □

### 4.3 Concluding the proof

In order to verify Conjecture 1, we need one more lemma, which is given below:

**Lemma 2.** *There exists a non-singular projectivity* $\kappa^\star$ *with* $\mathsf{n}_i\kappa^\star = \mathsf{N}_i^\star$ *for* $i = 1,\ldots,6$. *Therefore, the manipulator with platform anchor points* $\mathsf{n}_1,\ldots,\mathsf{n}_6$ *and base anchor points* $\mathsf{N}_1^\star,\ldots,\mathsf{N}_6^\star$ *is also a planar projective SG platform with an elliptic self-motion* $\mathscr{E}^\star$.

*Proof.* Due to Lemma 1 the points $\mathsf{N}_1^\star,\mathsf{N}_2^\star,\mathsf{N}_4^\star,\mathsf{N}_5^\star$ always form a quadrangle. Therefore, the mapping $\mathsf{n}_i \mapsto \mathsf{N}_i^\star$ for $i = 1,2,4,5$ uniquely defines a regular projectivity $\kappa^\star$. It can easily be seen by the collinearity properties of the anchor points, that also $\mathsf{n}_3\kappa^\star = \mathsf{N}_3^\star$ and $\mathsf{n}_6\kappa^\star = \mathsf{N}_6^\star$ hold.

Moreover, the elliptic self-motion $\mathscr{E}$ of the manipulator with platform anchor points $\mathsf{n}_1,\ldots,\mathsf{n}_6$ and base anchor points $\mathsf{N}_1,\ldots,\mathsf{N}_6$ is transmitted by the motion of the reguli $\mathscr{R}_1,\mathscr{R}_2,\mathscr{R}_3$ onto the manipulator with platform anchor points $\mathsf{n}_1,\ldots,\mathsf{n}_6$ and base anchor points $\mathsf{N}_1^\star,\ldots,\mathsf{N}_6^\star$. This resulting self-motion denoted by $\mathscr{E}^\star$ has to be elliptic, as a fixed point of the restriction of $\kappa^\star$ on $\mathsf{s} = \mathsf{s}\kappa^\star$ also has to be a fixed point of the restriction of $\kappa$ on $\mathsf{s} = \mathsf{s}\kappa$. As this would contradict our assumption that $\mathscr{E}$ is an elliptic self-motion, we are done. □

Now the proof of the conjecture can be closed by giving the construction for a special choice of the plane $\varepsilon$ (cf. Fig. 3b):

We consider any finite point $\mathsf{S} \in \mathsf{s} = \mathsf{s}\kappa$. This point spans together with the ideal points $\mathsf{f} \in \pi_\mathsf{m}$ and $\mathsf{F} \in \pi_\mathsf{M}$ (cf. Section 2.1) the plane $\alpha$.[2] Now we intersect $\alpha$ with a plane $\beta$, which contains $\mathsf{S}$ and is orthogonal to the direction $\mathsf{f}$. We denote the line of intersection by $\mathsf{t}$. Then we chose $\varepsilon$ as the plane spanned by $\mathsf{s} = \mathsf{s}\kappa$ and $\mathsf{t}$.

Due to Lemma 2, the resulting planar projective SG platform with platform anchor points $\mathsf{n}_1,\ldots,\mathsf{n}_6$ and base anchor points $\mathsf{N}_1^\star,\ldots,\mathsf{N}_6^\star$ possesses an elliptic self-motion $\mathscr{E}^\star$. According to the given construction, $\mathsf{f}\kappa^\star$ equals the ideal point of $\mathsf{t}$ and therefore $\mathscr{E}^\star$ is orthogonal. As planar projective SG platforms with such a self-motion do not exist (cf. Theorem 1), we end up with a contradiction. □

## 5 Conclusion

In this paper, we identified that the method based on the interaction of geometry and symbolic computation, which was used to prove Theorem 1, fails for solving the generalized problem formulated in Conjecture 1, due to the resulting computational complexity. By pure geometric reasonings, based on some historical results, we were able to verify Conjecture 1 by reducing the problem to the already solved one given in Theorem 1. This is a prime example for the fact that geometry is essential for solving advanced problems within the field of computational kinematics.

[2] Note, that $\mathsf{f} \in \mathsf{s}$ or $\mathsf{F} \in \mathsf{s}\kappa$ cannot hold as this yields $\mathsf{f} = \mathsf{F}$, a contradiction.

As the proof of Conjecture 1 also closes the study of planar projective SG platforms with self-motions, we can give the following main theorem under consideration of the results achieved in [10]:

**Theorem 4.** *A planar projective SG platform, which is not architecturally singular, can only have a self-motion if the projectivity is an affinity* $\mathbf{a}+\mathbf{A}\mathbf{x}$*, where the singular values* $s_1$ *and* $s_2$ *of the* $2\times 2$ *transformation matrix* $\mathbf{A}$ *with* $0 < s_1 \leq s_2$ *fulfill the condition* $s_1 \leq 1 \leq s_2$*. All one-parametric self-motions of these manipulators are circular translations. Moreover, the self-motion is a two-dimensional translation, if and only if, the platform and the base are congruent and all legs have equal length.*

**Acknowledgements** This research is supported by Grant No. I 408-N13 of the Austrian Science Fund FWF within the project "Flexible polyhedra and frameworks in different spaces", an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research.

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